An Investigation and Documentation of Social Choice Functions

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Abstract

Within this paper we develop a comprehensive and consistent index of voting system criteria in order to standardize the inconsistent notation used within social choice theory. Furthermore we demonstrate how one can utilize this resource to develop results which further inform us about the nature of social choice as a whole.

1 Introduction

In 2016 Donald Trump won the American presidency despite receiving nearly three million fewer votes than the leading candidate, Hillary Clinton. With occurrences like this many Americans have called for election reform so that the results of our elections more accurately reflect the preferences of America's population. In the modern world, the demand for fair and reliable methods of determining a group's consensus is ever-growing. If we want to do so, it is relatively obvious we ought to consider more than just each voter's first preference, but each voter's entire ranking of the existing options. In other words, we want methods that have a list of ballots as an input, and output the option(s) that best reflect the inputted ballots according to the method. We call these methods **social choice functions**. In this paper, we will go about defining a collection of various properties from a variety of different sources in an attempt to develop a consistent library of these properties. In addition, we will observe the multitude of relationships these properties have with each other which lead to the inevitable conclusion that no voting system can be optimally fair in every sense.

For a finite set of alternatives or candidates, $\mathcal{A}$ and a finite set of voters $\mathcal{V}$ where each voter has a ballot on which is an ordering of the candidates we define a preference schedule to be the tuple of all voter's preference ballots. A social choice function is any function from the set of all possible preference schedules, which we will call $\mathbb{P}(\mathcal{A}, \mathcal{V})$, to the set of non-empty subsets of $\mathcal{A}$. For example, **plurality without tiebreakers** would be the choice function that returns the smallest non-empty subset of $\mathcal{A}$ where no member of the subset has fewer first-place placements than any other candidate. A **dictatorship** is a social choice function that returns the first-place choice of a predetermined voter called the dictator. Another similarly 'unfair' social choice function is that of a monarchy which always returns the same unique candidate, in other words, if f is a monarchy for candidate $\mathrm{A} \in \mathcal{A}$ then $f(\mathbb{S}) = \{\mathrm{A}\}$ for all $\mathbb{S} \in \mathbb{P}(\mathcal{A}, \mathcal{V})$. Before moving forward, we introduce a couple of basic notations. For two preference schedules, $\mathbb{S}_1$ and $\mathbb{S}_2$ we write their join as $\mathbb{S}_1 + \mathbb{S}_2$, that is

$\mathbb{S}_1 + \mathbb{S}_2$ is the preference schedule consisting of all ballots cast in $\mathbb{S}_1$ and $\mathbb{S}_2$. The reason we use $+$ instead of $\cup$ is that $\mathbb{S}_1$ and $\mathbb{S}_2$ are not sets and referring to them as such, even indirectly, is misleading. Now, we will define quite a few criteria. They will be listed in alphabetical order for easy navigation.

2 Criteria

Definition 1: Anonymity Criterion

A social choice function satisfies the anonymity criterion if the result is always unchanged upon permutation of voters

In other words, a social choice function is anonymous if every voter is treated equally. A standard plurality election is anonymous while a dictatorship is clearly not anonymous as the dictator is the only voter who has any effect on the result.

We say that a candidate is a **Condorcet candidate** if they are preferred to each other candidate by a majority of voters, and we say that a candidate is an **Anti-Condorcet candidate** or a Condorcet loser if each other candidate is preferred to them by a majority of voters. With these terms in mind, we define the following criteria.

Definition 2: Anti-Condorcet Criterion

A social choice function satisfies the Anti-Condorcet criterion if an Anti-Condorcet candidate can never win. Sometimes called Condorcet loser criterion.

Definition 3: Condorcet Criterion

A social choice function satisfies the Condorcet criterion if when a Condorcet candidate exists, said Condorcet candidate will always be the unique winner of the election. Sometimes called Condorcet winner criterion.

One such method which satisfies both of these criteria is known as **Copeland's method** which is defined as follows. Each candidate is put in a head-to-head matchup against every other candidate. Each candidate games 1 point if they win, 0.5 if they tie, and 0 if they lose per matchup. The candidate(s) with the most points wins. It is easy to see why this method satisfies both the Anti-Condorcet criterion as there will always exist a candidate who gets a total point value greater than zero, and an anti-Condorcet candidate gets zero points under Copeland's method. The Condorcet criterion is also seen to hold as well because a Condorcet candidate will always have $n-1$ points, where n is the number of candidates, and at most one candidate can have $n-1$ points, as doing so requires that the candidate wins the head-to-head against all other $n-1$ candidates, something only a Condorcet candidate does by definition. We appropriately call methods that satisfy the Condorcet criterion like Copeland's Method Condorcet methods.

Definition 4: Decisive

A social choice function is decisive if the winner set is always a singleton.

Decisiveness is relatively simple, a social choice function is easily seen to be decisive if and only if it never allows ties. This criterion is not satisfied by methods like plurality without tiebreakers and Copeland's method but is satisfied by monarchy and dictatorship.

Definition 5: Homogeneity

A social choice function is homogeneous if the winner set depends only upon the proportion of each type of ballot.

Homogeneity essentially requires that if you combine arbitrarily many copies of a singular preference schedule into one larger preference schedule, the larger preference schedule will have the same result as the smaller one. While it may at first appear the homogeneity and anonymity are equivalent that is not the case. It is true that all homogeneous social choice functions are anonymous, we can define the following anonymous, yet non-homogeneous social choice function. First, eliminate all candidates who have less than three first-place placements on a ballot (if all candidates are eliminated, all candidates tie), then determine the winner via Copeland's method from the remaining candidates.

We say that a subset of the candidates constitutes a set of **clones** if no voter ranks a candidate outside the set of clones between any two clones. For example, in the below preference schedule candidates, C and D are clones as no voter ranks any candidate between candidates C and D.

A	**D**	B	E	**C**
E	**C**	E	**C**	**D**
B	A	**D**	**D**	E
C	B	**C**	A	B
D	E	A	B	A

Table 1: Clone candidates C and D on a preference schedule

Definition 6: Independence of Clones

The removal or addition of clone candidates who are not clones of candidates in the winner set does not change the outcome of the election.

Independence of clones is a rather restrictive yet sensical criterion that is satisfied by quite a few methods that many experts endorse. One such method is **Instant Runoff Voting** which is elegantly illustrated by the following flowchart courtesy of Wikipedia.

By last-place candidate, this diagram refers to the candidate(s) which have the smallest number of first-place votes.

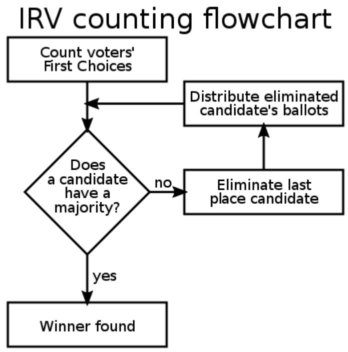

Figure 1: Instant Runoff Voting

Definition 7: Independence of Greatest Preference (IGP)

A social choice function is independent of greatest preference if the addition of a ballot with candidate A as the first choice to a preference schedule cannot change the result from candidate A winning to candidate A losing.

Many methods satisfy IGP including every method we have mentioned up until this point. However, one can quite easily conjure up methods that do not satisfy this criterion. For example, the **anti-dictatorship** does not satisfy IGP, it is defined to be the method that selects the last-place choice of a specific voter to be the winner, that voter being the eponymous anti-dictator.

Definition 8: Independence of Irrelevant Alternatives (IIA)

A social choice function is independent of irrelevant alternatives if the method that chooses candidate A to win over candidate B, in any profile where the same voters that preferred A over B still prefer A over B (and vice versa) results in B not winning.

IIA is likely one of the most, if not the most, discussed criteria on this list. It is incredibly controversial and has drastic consequences when combined with alone and when combined with other criteria. It is one of two criteria that together necessitate dictatorship, and very few systems satisfy it, two of which are dictatorship and anti-dictatorship.

Definition 9: Independence of Least Preference (ILP)

A social choice function is independent of least preference if the addition of a ballot with candidate A as the last choice to a preference schedule cannot change the result from candidate A losing to candidate A winning.

Like IGP, this criterion is satisfied relatively easily, with all voting systems previously mentioned, save the non-sensical anti-dictatorship, satisfying it.

For a given preference schedule, we say that a set is S is *dominating* if it is a non-empty subset of the set of candidates, $\mathcal{A}$ such that for any head-to-head matchup between a member of S and a member of $\mathcal{A} \setminus S$ the member of S always wins. The smallest dominating set is the *Smith set*.

Definition 10: Independence of Smith-dominated Alternatives (ISDA)

A social choice function is independent of Smith-dominated alternatives if removing candidates, not in the Smith set has no impact on the result.

One can easily see that any method satisfying ISDA also satisfies the previously mentioned Condorcet criterion, including the **Kemeny-Young method**. This method looks at each possible ranking of the set of candidates, and for each pair of candidates, the number of ballots that rank them opposite of the given ranking is cumulatively added to a final distance score. The winner is the candidate(s) who ranks at the top of the rankings with the least distance.

For a social choice function f, we say that a voter is a *liberal* if there exists a pair of candidates A and B such that for any preference schedule $\mathbb{S}_1$ where the voter ranks candidate A above candidate B then $\text{A} \in f(\mathbb{S}_1)$ and for any preference schedule $\mathbb{S}_2$ where the voter ranks candidate B over candidate A we have that $\text{B} \in f(\mathbb{S}_2)$. In a more intuitive sense, a voter is a liberal if there is a pair of alternatives for which their opinion is always reflected in the results of the election. The reason we call such voters liberals is due to the classical liberal idea of one always having the freedom to choose between their options.

Definition 11: Liberal Criterion

A social choice function satisfies the liberal criterion if every voter is a liberal.

This criterion is actually quite restrictive, especially for social choice functions that attempt to determine a sole winner most of the time. The liberal criterion is more applicable when applied to social choice functions in which the various options are not necessarily mutually exclusive by nature. For example, consider a pair of bookstore owners deciding which books they want to add to their stock from a set of 100 books when they only have a budget for 20 books. Let Owner A be an expert in true crime, and let Owner B be an expert in historical fiction. Now, assume of the 100 books there are two true crime books and two historical fiction books. Due to each owner's expertise in their respective subjects, Owner A's preferred true crime book as well as Owner B's preferred historical fiction book will be added to the bookstore's shelves. As for the other 18 books, we can just select the 18 books (or potentially less in the case of a tie) with the highest rank by some other social choice function that are not true crime or historical fiction. This carefully constructed social choice function does quite clearly satisfy the liberal criterion as Owner A is a liberal for the pair of true crime books and Owner B is a liberal for the pair of historical fiction books. As we will see in the coming sections of this report the restrictive nature of this criterion is quite demonstratable.

With this relatively controversial criterion defined, we move to what is likely one of the least controversial of all criteria.

Definition 12: Majority Criterion

A social choice function satisfies the majority criterion if a candidate receiving the majority of first-place votes always wins.

The Majority criterion is satisfied by nearly every social choice function attempting to be fair and is not satisfied by every intentionally unfair method we have defined up until this point. However, there exist methods like the **Borda Count** which is an attempt at a fair social choice function that does not satisfy the majority criterion. The Borda count is what we call a **positional** voting system in which candidates receive points based on their ranking on each ballot with the winner being the candidate with the greatest number of points. In the Borda count the number of points given for rank r is equal to $n - r$ where n is the number of candidates. One can easily show that the Borda Count does not satisfy the majority criterion by examining the following preference schedule in which A receives the majority of first-place votes, but B wins under the Borda count.

A	A	A	E	E
B	B	B	B	B
C	C	C	C	C
D	D	D	D	D
E	E	E	A	A

Table 2: Example of Borda count's majority non-compliance.

Definition 13: Majority Loser Criterion

A social choice function satisfies the majority loser criterion if a candidate that receives the majority of last-place votes always loses.

Despite how similar the majority loser and the majority winner are as criteria the Borda Count actually satisfies the majority loser criterion, and this can be proven as follows.

Let f be a Borda count social choice function for v voters and a alternatives. Note that each ballot ranking a alternatives adds $a(a-1)/2$ points to the total number of points shared between all candidates, and because there are v voters the sum of all candidates' scores will be $va(a-1)/2$. By the pigeonhole principle there exists at least one candidate with at least $v(a-1)/2$ points, and thus the winner will have at least $v(a-1)/2$ points. Now, assume that there is some majority loser candidate, note that this candidate must receive less than $v(a-1)/2$ points, and thus cannot be the winner.

Definition 14: Manipulability

We say that a social choice function is manipulable if a voter can change their vote to reflect a preference order different from their own to receive a more favorable outcome.

Manipulability (more specifically non-manipulability), while relatively simple in concept, is a rather complex criterion with deep-reaching consequences demonstrated by the Gibbard-Satterthwaite Theorem. One can see that all of dictatorship and are monarchy non-manipulable, however, through various examples, it can be seen that every other method described up until now is manipulable.

Definition 15: Monotonicity Criterion

A social choice function satisfies the monotonicity criterion if, in a scenario where candidate A wins, somebody changes their vote to rank A higher than before without making any other changes, A remains the winner.

Monotonicity is another property that seemingly should be on every social choice function intending to be fair, it is satisfied by the Borda Count, dictatorship, but is not satisfied by instant runoff voting. This can be seen by examining the following two preference schedules in which B is the unique winner under IRV originally, but upon making no changes save for moving B up on the ballot, A becomes the unique winner.

A	A	A	A	B	B	B	B	B	C	C	C	C
C	C	C	C	A	A	A	A	A	A	A	A	A
B	B	B	B	C	C	C	C	C	B	B	B	B

Table 3: Preference schedule where B is the winning candidate under IRV.

A	A	A	A	B	B	B	B	B	**B**	C	C	C
C	C	C	C	A	A	A	A	A	**C**	A	A	A
B	B	B	B	C	C	C	C	C	**A**	B	B	B

Table 4: Modified preference schedule where A is the winning candidate under IRV after shifting B upwards on the bold ballot.

Before seeing this, one would likely see nothing potentially unfair regarding instant runoff voting, which is why it is important to examine them in detail with criteria like the above.

Definition 16: Mutual Majority Criterion

A social choice function satisfies the mutual majority criterion if for all sets S where a majority of voters prefer every candidate in S to every candidate outside S the winner must be in S. We call S a mutual-majority-preferred set.

Note that despite the similarity between the definitions of mutual-majority-preferred set and type dominating set, they are in fact different. Specifically, a mutual-majority-preferred set is a type of dominating set. A dominating set only requires that each candidate in the set is preferred to candidates outside the set by some majority of voters whereas a mutual-majority-preferred set requires that there is a coalition of voters that form a majority such that every voter in the coalition ranks every candidate inside the mutual-majority-preferred set above every candidate outside the mutual-majority-preferred set.

Mutual Majority is satisfied by IRV and Kemeny-Young but is not satisfied by Borda Count. Another example of a voting system that satisfies the mutual majority criterion is called **Bucklin voting** which determines the winner as follows. If any candidate has a majority (greater than $v/2$ where v is the number of voters) of the votes, then that candidate is declared the winner, otherwise, add the second-place choices to the first-place choices, if there is still no candidate with a majority of votes add the third place votes to the first-place votes, and so on until one or more candidate with more than $v/2$ votes exist and declare those candidate(s) the winner(s).

Definition 17: Neutrality Criterion

A social choice function satisfies the neutrality criterion if relabeling of the candidates results in the winner being appropriately changed. That is, if a winner is relabeled to candidate X, then candidate X wins, and if a loser is relabeled to candidate Y, then candidate Y loses.

The neutrality criterion is likely the least controversial of all criteria as it essentially just states that all candidates are treated equally with no preferential treatment being directed towards any one of the candidates. Neutrality and anonymity together allow a social choice function to both not inherently prefer any one candidate over another as well as not give any one voter more power than another. Despite this, in combination neutrality and anonymity prohibit another criterion, namely decisiveness. To see why this is, consider the following preference $\mathbb{S}$ schedule under some social choice function that is anonymous, decisive, and neutral.

A	B
B	A

Table 5: Preference schedule $\mathbb{S}$

Because the social choice function is decisive the winner set cannot be $\{A, B\}$. Without loss of generality, assume that the winner set consists only of A. If this were the case, neutrality would state that the winner of the below preference schedule, $\mathbb{S}'$ is B, however, anonymity would state that the winner is A, a contradiction to the original assumption that an anonymous, decisive, and neutral social choice function exists.

B	A
A	B

Table 6: Preference schedule $\mathbb{S}'$

Thus, we have the following proposition.

Proposition 18

There is no social choice function that is anonymous, decisive, and neutral.

In some literature, a social choice function that is both anonymous and neutral is said the be **symmetric**.

With this, one may want a weaker form of neutrality for their social choice function in lieu of its required absence with anonymity and decisiveness. Specifically, you may want to ensure that there is no candidate that never wins. For this, we define non-imposition.

Definition 19: Non-Imposed

A social choice function is Non-Imposed if for every candidate there exists at least one preference schedule for which that candidate is a member of the winner set.

Every voting system we have mentioned above other than monarchy is non-imposed. Later on, in this paper, we will explore an implication of the Gibbard-Satterthwaite Theorem which implies that all social choice functions of a certain kind are imposed.

When determining which of anonymity, decisiveness, and neutrality to relax and decisiveness is selected, it may be replaced with the ability to ensure that breaking a tie is possible. This property of a social choice function is called Resolvability and is defined formally below.

Definition 20: Resolvability

A social choice function is resolvable if, for every (possibly tied) winner in a result, there exists a ballot that, when added, results in the previously mentioned winner being a unique winner.

While it is vacuously true that all decisiveness social choice functions are resolvable, the reverse is not true. For an example of this in action, we look towards the plurality without tiebreakers method defined at the start of this report. Clearly, whenever there is a tie between two or more candidates under plurality without tiebreakers the addition of a ballot that has one of those tied candidates ranked first will break the tie and cause one of the previously tied candidates to become the unique winner, and thus plurality without tiebreakers is resolvable.

For some preference schedule $\mathbb{S}$, let $-\mathbb{S}$ be the preference schedule where every ballot is reversed. With this, we define the following criterion.

Definition 21: Reversal Symmetry

A social choice function f satisfies reversal symmetry if for any preference schedule $\mathbb{S}$ where $f(\mathbb{S}) = \{\text{A}\}$ we have that $A \notin f(-\mathbb{S})$.

Reversal symmetry is an attempt to ensure that if a candidate wins that the social choice function would interpret the reverse scenario as a loss for the candidate. However, reversal symmetry by itself is not meant to indicate the fairness of a social choice function as social choice functions that reward less preferred candidates consistently also are often reversal symmetric. Specifically, we can see that if f is reversal symmetric the social choice function g defined by $g(\mathbb{S}) = f(-\mathbb{S})$ is reversal symmetric.

Definition 22: Pareto Criterion

A social choice function satisfies the Pareto criterion if a candidate who unanimously loses to another candidate in a head-to-head match-up never wins.

Despite this criterion being one that most would require their social choice function to have, the only social choice function on three or more candidates that satisfies IIA and Pareto is a dictatorship as per Arrow's Theorem. [1].

Definition 23: Participation Criterion

A social choice function satisfies the participation criterion if the addition of a ballot to where candidate A is preferred over candidate B will not change the winner set such that it no longer contains candidate A, but does now contain candidate B.

The participation criterion is satisfied by both plurality and Borda count however is mutually exclusive with every Condorcet method, which is proven in [5].

Definition 24: Separability Criterion

A social choice function f is separable if for any two preference schedules $\mathbb{S}_1$ and $\mathbb{S}_2$ we have that if $f(\mathbb{S}_1) \cap f(\mathbb{S}_2) \neq \emptyset$ then $f(\mathbb{S}_1) \cap f(\mathbb{S}_2) = f(\mathbb{S}_1 + \mathbb{S}_2)$. Sometimes called consistency (though, this name is too vague for my liking).

When separability is satisfied, it is very common that participation is satisfied. In addition, we actually have a complete classification of neutral separable social choice functions courtesy of [4].

Theorem 25: Classification of Separable Social Choice Functions [4]

A neutral social choice function is separable if and only if it is a positional voting system.

From this, it follows that both the Borda count and plurality are separate. We also have that the **Pick Two** method is separable. The pick-two method sums the number of first-place votes and second-place votes each candidate receives with the winner being the candidate(s) with the greatest number of summed votes. This is seen to be positional as a candidate gets one point for first and second place, and zero for every other placement.

Definition 26: Smith Criterion

A social choice function satisfies the Smith criterion if the winner is always in the smith set.

This criterion is automatically satisfied by any social choice function satisfying ISDA, and the Smith criterion implies all of the Condorcet, anti-Condorcet, and majority criteria. One example of a social choice function satisfying the Smith criterion is **Nanson's method**, which determines a winner as follows. First, calculate a score for each candidate like done in the Borda Count, then every candidate who's Borda score is below the average of all the candidates' scores are eliminated. This process is repeated on the new candidate set as if the eliminated candidates never existed until every remaining candidate has the same Borda score, and those candidate(s) are deemed the winner(s).

Definition 27: Top Criterion

A social choice function satisfies the top criterion a candidate cannot be in the winner set unless they receive at least one first-place vote.

The top criterion is one that you may want for your social choice function if you want to ensure that the winner is never nobody's first choice. Despite how appealing this may sound it is actually impossible for a social choice function to satisfy both the Condorcet criterion and the top criterion. We can actually prove this quite easily by examining the following preference schedule where A is a Condorcet candidate that also has no first-place votes.

E	E	C	B	B
A	A	A	A	A
B	B	E	E	E
C	C	B	C	C
D	D	D	D	D

Definition 28: Unanimity Criterion

A social choice function satisfies the unanimity criterion if a candidate receiving every first-place vote always wins.

Unanimity is a relatively simple criterion that requires little explanation. However, I would like to note the easy observation that Pareto $\implies$ Unanimity $\implies$ Non-imposition. Furthermore, every Condorcet method satisfies Unanimity.

In some literature, Condorcet candidates are permitted to draw in head-to-head match-ups with other candidates. However, here we do not allow such a thing, and instead introduce the idea of a *Weak Condorcet* candidate, one which does not lose any head-to-head match-ups. Clearly, every Condorcet candidate is a Weak Condorcet Candidate but not vice-versa.

Definition 29: Weak Condorcet Criterion

A social choice function satisfies the Weak Condorcet criterion if the non-empty set of all Weak Condorcet candidates is equal to the winner set.

Throughout my research, I would often stumble upon the idea that no separable social choice function can be Condorcet-compliant. However, the proof found in [3] under careful analysis, proves instead that a social choice function cannot be weakly Condorcet and separable. While it is true that all weakly Condorcet social choice functions are Condorcet, there are Condorcet social choice functions that are not weakly Condorcet. One example is the social choice function that selects the Condorcet candidate as the unique winner if they exist, but all candidates tie otherwise. However, this method is not separable either, and so this begs the question: "Are there any social choice functions that are both Condorcet and Separable?"

While the previously mentioned 'all-ties unless Condorcet' method is not separable it is, in fact, homogeneous, a weaker form of separability which is also incompatible with the weak Condorcet criterion as per [3].

Our final criteria relies on the idea of the $*$-transform which we borrow from an exercise in [6].

Let P be a social choice function. We call P^* the $*$ (star) transform of P. P^* is the social choice function defined as follows.

1. Take every voter's preference order and reverse it.
2. with this new preference schedule, determine the winner, and eliminate said winner from the candidate list
3. repeat the above two steps with the new candidate list until every candidate would be eliminated

Definition 30: $*$-Reflexive

We say that a social choice function is $*$ reflexive if $P = P^*$.

Like reversal symmetry, $*$-reflexivity is an attempt to ensure that the reverse of a preference schedule does not result in a positive outcome for the original winner. Examples of reversal symmetric social choice functions include the **all-ties** method, which always returns the entire set of alternatives, and dictatorship.

With this plethora of criteria under our belt, we move onto investigating the various relationships between them, specifically we want to see how restrictive these criteria are when compounded upon each other.

3 Impossibility

Within social choice theory, one of the most intriguing types of results are *impossibility theorems*. An impossibility theorem states that a certain set of conditions cannot all be true at once in regard to social choice functions. The most famous impossibility theorem is that of Arrow, which we mentioned earlier. The formal statement is the following.

Theorem 31: Arrow's Impossibility Theorem [1]

A social choice function on three or more candidates cannot have all of the following properties.

- Pareto.
- IIA.
- Non-Dictatorship.

The significance of this Theorem should not be understated as it implies that only the dictatorship can satisfy the seemingly unassuming precedents of Pareto and IIA, which is not only a very small subset of choice functions, but it is, by common sense standards, atrocious. Arrow's impossibility theorem was the first of its kind, and it paved the way for multiple other authors to develop similar theorems including the next most famous impossibility theorem is credited both to Gibbard and Satterthwaite and is the following.

Theorem 32: Gibbard-Satterthwaite Theorem [7]

A social choice function cannot have all of the following properties.

- Non-Manipulability.
- Decisive
- Three or more alternatives are capable of winning.
- Non-Dictatorship

Like Arrow's theorem, the Gibbard-Satterthwaite theorem demonstrates that only a dictatorship can satisfy a pair of seemingly basic requirements for a social choice function. To better understand non-manipulability let us examine the following preference schedule under the Borda count, a social choice function that satisfies both of the latter two conditions in the Gibbard-Satterthwaite theorem. Under Borda

v_1	v_2
A	B
B	C
C	A

count, the winner of the above election is seen to be B. Suppose the above preference schedule reflects each voter's actual preferences, however, assume that instead v_1 votes dishonestly with the ballot (A, C, B). This new election results in a three-way tie which includes candidate A, the favorite of v_1, in the winner set. This means that v_1 prefers the modified outcome despite them submitting a ballot they agree with less, demonstrating the manipulability of the Borda count. Gibbard-Satterthwaite says that every voting system other than dictatorships and ones which limit the results to two or fewer candidates must be susceptible to the strategic voting demonstrated above.

Another theorem in the vein of Arrow and Gibbard-Satterthwaite was proven by economist Amartya Sen and is somewhat whimsically titled *the liberal paradox.* The statement for this "paradox" is as follows.

Theorem 33: Liberal Paradox [8]

A social choice function cannot have both the liberal and Pareto criteria.

We can see this result in action by examining our bookstore example from earlier. Assume that our bookstore social choice function is Paretian and examine the following preference schedule where TC_1 and TC_2 are the true crime novels, and X is some other book.

A	B
X	X
TC_1	TC_1
TC_2	TC_2
⋮	⋮

Because Owner A is a liberal for TC_1 and TC_2, and because Owner A ranks TC_1 over TC_2 we must have that TC_1 is in the winner set. However, if the social choice function were to be Paretian TC_1 could not be in the winner set as X is unanimously preferred to TC_1.

There are a multitude of other impossibility theorems out there like the Dugan-Schwartz Theorem [10] which is too complex for the scope of the report, and the Balinski-Young Theorem [11] which relates to apportionment, which is also outside the scope of this report.

4 Combining Criteria

In this section, we will demonstrate how one can develop by regarding the combination of various criteria. First, we will show that the novel concept of $*$-reflexivity is implied by a triplet of established criteria. For this proof, we introduce the following notation. For a preference schedule $\mathbb{S}$ for a set of alternatives $\mathcal{A}$ and some $\text{A} \in \mathcal{A}$ we define $\mathbb{S} - \text{A}$ to be the preference schedule to be the unique preference schedule for $\mathcal{A} \setminus \{\text{A}\}$ for which the preferences between all candidates in $\mathcal{A} \setminus \{\text{A}\}$ are the same in $\mathbb{S}$ and $\mathbb{S} - \text{A}$. For an example, see the below preference schedule.

A	A	B	C
B	B	D	A
C	C	A	B
D	D	C	D

Table 7: Preference schedule $\mathbb{S}$.

And now observe the same preference schedule with a candidate removed.

A	A	D	C
C	C	A	A
D	D	C	D

Table 8: Preference schedule $\mathbb{S} - \text{B}$.

Proposition 34

Any social choice function that satisfies reversal symmetry, decisiveness, and IIA must be $*$-reflexive.

Proof. Let f be a social choice function satisfying reversal symmetry, decisiveness, and IIA. Since f is decisive, it has a unique winner for every preference schedule. Let $\mathbb{S}$ be some arbitrary preference schedule, and let $f(\mathbb{S}) = \{\text{A}\}$. Let $-\mathbb{S}$ be the preference schedule $\mathbb{S}$ but with all ballots reversed. Because f satisfies reversal symmetry, it is easily seen that $\text{A} \notin f(-\mathbb{S})$ Without loss of generality, let $f(-\mathbb{S}) = \text{B}$. Thus, some non-A-alternative is eliminated first when determining $f^*(\mathbb{S})$. Furthermore, upon removing B from the set of alternatives, from IIA we know that $f(\mathbb{S} - \text{B}) = A$, and the above steps can be repeated to show that at each step A remains the unique winner and thus $f^*(\mathbb{S}) = f(\mathbb{S}) = \{\text{A}\}$ making f $*$-reflexive. ∎

This can be seen through various examples including the dictatorship method, which is seen to be decisive, reversal symmetric, and independent of irrelevant alternatives. Thus, it follows from the above proposition that the dictatorship satisfies $*$-reflexivity. Indeed, it is quickly observed that the dictatorship is $*$-reflexive. This proposition is

One can see that not all $*$-reflexive social choice functions are included above through the example of all-ties. In order to fully classify which social choice functions are $*$-reflexive we must determine which properties are inherent to $*$-reflexive functions. Thus, I offer the following problem

Problem 35

Classify all $*$-reflexive social choice functions.

Proposition 36

If f is a social choice function on a set of candidates $\mathcal{A}$ and set of v voters $\mathcal{V}$ satisfying IIA, Decisiveness, Separability, and Non-Manipulability then there must exist a pair of alternatives $\mathrm{X}, \mathrm{Y} \in \mathcal{A}$ for which the following holds.

- There exists some real number r such that $0 \leq r \leq 1/2$.
- Candidate X will be the unique output of f if at least rv voters rank Candidate X above candidate Y.
- Candidate Y will be the unique output of f otherwise.

Proof. Let f be a social choice function satisfying the above criteria. By Gibbard-Satterthwaite we know that f cannot have more than two valid outcomes as f is decisive, non-manipulable, and non-dictatorial (from separability). Furthermore, because f is decisive these outcomes are both singletons. Let A and B be the candidates who are capable of winning. Furthermore, because f satisfies IIA and anonymity (from separability) we know that the winner of any election is reliant solely upon the number of voters who prefer candidate A over candidate B compared to the number of voters who prefer Candidate B over candidate A.

Consider the preference schedules $\mathbb{S}_0$ consisting solely of a ballot of the form $(\mathrm{A}, \mathrm{B}, \dots)$, and $\mathbb{S}_1$ consisting solely of a ballot of the form $(\mathrm{B}, \mathrm{A}, \dots)$. If $f(\mathbb{S}_0) = \mathrm{B}$ or $f(\mathbb{S}_1) = \mathrm{A}$ then it follows from IIA and separability that $f(\mathbb{S}) = \{\mathrm{B}\}$ or $f(\mathbb{S}) = \{\mathrm{A}\}$ respectively for all $\mathbb{S} \in \mathbb{P}(\mathcal{A}, \mathcal{V})$ and thus f is a monarchy, without loss of generality, if one of the two candidates were to be a monarch, it will be X.

The other possible scenario is that $f(\mathbb{S}_0) = \{\mathrm{A}\}$ and $f(\mathbb{S}_1) = \{\mathrm{B}\}$. In this scenario, let $\mathbb{S}_2 = \mathbb{S}_0 + \mathbb{S}_1$ and without loss of generality let $f(\mathbb{S}_2) = \{\mathrm{A}\}$. From separability and IIA, it is seen that A wins in any preference schedule for which the number of voters who prefer candidate A over candidate B is greater than or equal to the number of voters who prefer candidate B over candidate A.

Furthermore, because non-manipulability implies monotonicity (See Appendix, Proposition 38), and because separability implies homogeneity. Collectively, f has the following properties.

1. Only X or Y can win, and they cannot tie.
2. All ballots that rank X above Y are treated the same, and all ballots that rank Y above X are treated the same.
3. The election is monotone.
4. Candidate X wins if they receive half or more of the votes.
5. The proportion of each type of ballot is enough to discern a result. (Homogeneity)

From these facts, it is seen that we can define this function by three parameters. Candidate X: The advantaged candidate, Candidate Y: The disadvantaged candidate, and r, X's target proportion. The function is as follows, If X is ranked higher than Y by a proportion of the voters greater than or equal to r then X is the unique winner. If X does not achieve this, Y is the unique winner. Furthermore, because we know X to win at a tie, r can be no greater than 1/2, and because it is a proportion, it cannot be less than zero. Furthermore, it is seen that when $r = 0$, we have our monarchy, which also can meet the four criteria as described above. These are the exact condition of our claim, and thus it is proven. □

As mentioned in the proof, it is seen that when $r = 0$, the above method is a monarchy for candidate X. Indeed one can easily observe that a monarchy is IIA, Decisive, Separable, and Non-Manipulable. We can also show that the $r = 1/2$ case satisfies the four conditions as well. Non-Manipulability, IIA, and Decisiveness are easily seen to be satisfied for all non-zero r.

Furthermore, we can prove the method is separable as follows. Let g be the social choice function described in Proposition 36 for a fixed r, and fixed candidates X and Y. Let r_1 and r_2 be the respective proportion of voters in preference schedules $\mathbb{S}_1$ and $\mathbb{S}_2$ that prefer Candidate X to Candidate Y. Furthermore, let v_1 and v_2 be the number of ballots in $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. If $g(\mathbb{S}_1) \neq g(\mathbb{S}_2)$, separability implies nothing about the concatenation of preference schedules with an empty winner set intersect. First, note that the proportion of voters that prefer voters that prefer X over Y in $\mathbb{S}_1 + \mathbb{S}_2$ is $\frac{r_1v_1+r_2v_2}{v_1+v_2}$. Now, consider when $g(\mathbb{S}_1) = g(\mathbb{S}_2) = \{\text{X}\}$ we must have that $r_1, r_2 \geq r$. Furthermore, it is seen that $\frac{r_1v_1+r_2v_2}{v_1+v_2} \geq \frac{rv_1+rv_2}{v_1+v_2} = r$ and thus $g(\mathbb{S}_1 + \mathbb{S}_2) = \{X\}$ giving us that $g(\mathbb{S}_1) = g(\mathbb{S}_2) = \{\text{X}\} \implies g(\mathbb{S}_1 + \mathbb{S}_2) = \{\text{X}\}$. . If $g(\mathbb{S}_1) = g(\mathbb{S}_2)$ the same process is done to show that $g(\mathbb{S}_1) = g(\mathbb{S}_2) = \{\text{Y}\} \implies g(\mathbb{S}_1 + \mathbb{S}_2) = \{\text{Y}\}$. With this, it is easily seen that g is separable. Thus, the proposition's implications go both ways.

5 Conclusion

But what do these propositions tell us? Well, the result on $*$-reflexivity demonstrates the interconnected nature of all basically all of the defined criteria. Despite $*$-reflexivity is a novel concept it is an attempt to formalize the idea fairness into rules that a social choice function must follow. Because of this, it is inevitable that one or more criterion occurring will imply the occurrence of another. On the other hand Proposition 36, while not as far reaching or impressive as the impossibility theorems, it does provide us with insight into the nature of the criteria. First, we see the nature of how restrictive these criteria really are, to see this, consider the following.

Proposition 37

For a system with c candidates and v voters there are

$$(2^c - 1)^{((c!)^v)}$$

social choice functions.

Proof. Each voter can submit a total of $c!$ different preference schedules, and thus our social choice function takes the one of $(c!)^v$ different preference schedules, and outputs one of $2^c - 1$ different winner sets. Thus there are

$$(2^c - 1)^{((c!)^v)}$$

social choice functions. □

So, for three candidates and three voters there is more than $3.476 \cdot 10^{182}$ social choice functions. However, if we count the number of social choice functions on three voters and three candidates that are all of IIA, Decisive, Separable, and Non-Manipulable by Proposition 36 we get that there are a mere *nine* social choice functions. Even worse, Arrow's Theorem tells us that every Pareto and IIA social choice function must be a dictatorship and so on three voters and three candidates there only *three* social choice functions that are both Paretian and IIA.

This brings me to our next point, due to the restrictive nature of these criteria, despite the fact that we may want a voting system that satisfies all of them that is simply impossible, and we are going to have to cut our losses somewhere. Examples of this can be seen in the real world, for example in the United States our elections are indeed decisive and neutral, however by Proposition 18 this means that the United States Presidential elections are not anonymous, and this is easily seen to be true! Even from a voting power standpoint, Wyoming voters share three electoral college members among their population of less than 600,000. Conversely, there are about 39 million Californians, and they have 55 electors. Thus, assuming the proportion of voting population to population proper is equal in both states, a Wyoming voter has over three times more representation in the presidential race than a Californian. This is a clear violation of anonymity, and Americans have been living with it for the entire history of the country. While it is likely due to a combination of reasons, the fact that our voting system is non-anonymous is necessary from the fact that we always want to ensure a candidate is elected, and the fact that it would be improper to treat one candidate favorably. Thus, it is seen through these criteria and their combinations that we can never hope to develop an ultimately fair way to determine a groups preference, but that will not stop us from trying.

A Appendix: Simple Implications

In this appendix, we produce a small collection of simpler implications that I wished to have documented but did not want to disrupt the flow of the main report, however, I would still like to include them.

Proposition 38

Non-Manipulability $\implies$ Monotone

Proof. To prove this, it suffices to show that to non-manipulable and non-monotone social choice function can exist. Let us assume the precedent that f is non-manipulable and non-monotone, and thus it is possible for any voter to raise a winning candidate one spot on their ballot such that they become a losing candidate. Let $\mathbb{S}_1$ be a preference schedule with the property that $\text{A} \in f(\mathbb{S}_1)$, and let $\mathbb{S}_2$ be a preference schedule identical to $\mathbb{S}_1$ in every way save that one specific voter has raised the ranking of A by one while preserving the relative ranking of all other candidates with the property that $\text{A} \notin f(\mathbb{S}_2)$.

Now, note that if $\mathbb{S}_2$ is an honest ballot that implies that the $f(\mathbb{S}_1) = \{\text{A}\}$ is a less preferred outcome than $f(\mathbb{S}_2)$ for our voter, and thus there is a candidate in $f(\mathbb{S}_2)$ ranked more preferably than A by the focused voter on $\mathbb{S}_2$. However, if $\mathbb{S}_2$ is dishonest then it is implied that the honest submission of the ballot in $\mathbb{S}_1$ giving $f(\mathbb{S}_1) = \{\text{A}\}$ is a more preferred outcome for our voter by non-manipulability, and thus $f(\mathbb{S}_2)$ cannot include any candidates ranked higher than A by our voter on $\mathbb{S}_1$. However, because our social choice function has no way of telling whether a ballot is honest or not both these cases must be accounted for in the social choice functions output. Now note that every candidate ranked higher than A by our voter on $\mathbb{S}_1$ is ranked higher than A by our voter on $\mathbb{S}_2$, meaning both of the above realities cannot be accounted for and thus we have a contradiction. This implies that our original assumption of a non-manipulable and non-monotone social choice function existing is false, proving our claim.

□

Proposition 39

Weak Condorcet $\implies \neg$ Decisive

Proof. In the below preference schedule, both Candidates A and B are weak Condorcet candidates, and thus they both must be in the winner set of any weak Condorcet method, and thus such a method cannot be decisive.

A	B
B	A

□

Proposition 40

- Mutual Majority $\implies$ Majority
- Mutual Majority $\implies$ Majority Loser

Proof. It is seen the majority criterion is a special case of the mutual majority criterion, namely, when the mutual-majority-preferred set is a singleton.

Furthermore, the majority loser criterion is the special case of the mutual majority criterion, specifically when the mutual-majority-preferred set consists of all but one of the candidates. □

We also will list a few implications which follow from the definitions easily. Some of these were mentioned earlier in the criteria section, however for the sake of collecting as many known implications as possible in one place, I will list them again here along with reasoning.

Proposition 41

- Weak Condorcet $\implies$ Condorcet
- Pareto $\implies$ Unanimity $\implies$ Non-Imposition
- ISDA $\implies$ Smith $\implies$ Condorcet
- Neutral $\implies$ Non-Imposed
- Separable $\implies$ Homogeneous $\implies$ Anonymous

Proof. The fact that Weak Condorcet implies Condorcet is trivial as every Condorcet candidate is also a weak Condorcet candidate.

It is easily seen that Pareto implies unanimity as placing the same candidate at the top of every preference schedule results in each other candidate being unanimously preferred over, and thus not electable by Pareto. Furthermore, unanimity implies non-imposition as regardless of which candidate is unanimously placed first, they will win.

ISDA implying Smith is trivial as ISDA requires that eliminating candidates not in the Smith set does not change the result of the election, and so no winning candidates can be outside of the smith set, as that would imply that eliminating a winning candidate does not change the result, which cannot be. It is also seen that Smith implies the Condorcet criterion as it is easily seen to be true that a set consisting of only a Condorcet candidate is a Smith set.

For a social choice function to be neutral we must have every candidate treated equally, that means if any one candidate is capable of winning, then every candidate is capable of winning, and since it is impossible to have an empty winner set, there is always at least one candidate capable of winning, and thus we have that every candidate is capable of winning, and this is non-imposition.

Homogeneity is easily seen to be the special case of separability when $\mathbb{S}_1 = \mathbb{S}_2$, and if only the proportion of each type of ballot matters when determining the result, then it is impossible for one voter to be treated differently from another, and thus homogeneity implies anonymity. □

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