Challenge Problems 5

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Problem Index

Problem	1a	1b	1c	2a	2b	2c	3ab	3c	4	5a	5
Difficulty/10	4	4	8	1	2	7	1	6	7	1	9
Category	NT	NT	NT	CO	CO	CO	CO	CO	NT	GT	GT

Key:

- NT: Number Theory
- CO: Combinatorics
- GT: Game Theory

1. Consider the function $a : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as

$$a(m, n) = \sum_{i=1}^{m} \prod_{j=1}^{i} j^n.$$

   (a) Prove that $a(m, n)$ is never prime for odd n.
   (Hint: Prove that this number is always divisible by three for $m \neq 1$.)

   (b) Prove that $a(m, 6k + 3)$ is divisible by 9 for all $k \geq 0$ and $m \geq 2$.

   (c) Prove the following statement.

   Claim. *For every positive integer s, there are an infinite number of positive integer values of n for which $a(m, n)$ is divisible by 3^s for all $m \geq 2$.*

2. Let $b(c, r)$ be the maximum number of different pieces a checkers king can capture in one turn on a c by r board.

   (a) Determine the value $b(8, 8)$.

   (b) Determine all pairs of c and r for which $b(c, r) \in \{0, 1, 2, 3\}$.

   (c) Find a formula for $b(c, r)$.

3. At a very busy restaurant a large party of people call in asking to reserve a table. Within this party there are n families with size $A = a_1, a_2, \ldots, a_n$. When deciding seating arrangements each of these families is to be seated together, in other words for each family there exists an interval of seats that contains exactly every member of that family and nobody else. Each seat is marked with a number, and thus even if two table arrangements are equivalent up to reflection, rotation, or other symmetries they count as separate arrangements.

   (a) Let $c_{\leftrightarrow}(A)$ be the number of different seating arrangements for a family sequence A in a row of seats containing exactly enough seats to seat each person.

i. Let $A = \underbrace{2, 2, \ldots, 2}_{\text{Eight 2's}}$, determine $c_{\leftrightarrow}(A)$.

ii. Find a formula for $c_{\leftrightarrow}(A)$ if A is exactly n copies of m.

iii. Find a formula for $c_{\leftrightarrow}(A)$ with arbitrary A.

(b) Let $c_{\circlearrowright}(A)$ be the number of different seating arrangements for a family sequence A on a round table of exactly enough seats to seat each person.

i. Let $A = \underbrace{2, 2, \ldots, 2}_{\text{Eight 2's}}$, determine $c_{\circlearrowright}(A)$.

ii. Find a formula for $c_{\circlearrowright}(A)$ if A is exactly n copies of m.

iii. Find a formula for $c_{\circlearrowright}(A)$ with arbitrary A.

(c) Let $c_{\infty}(A)$ be the number of different seating arrangements for a family sequence A on a lemniscate-shaped table with exactly one seat at the intersection as displayed in the figure below.

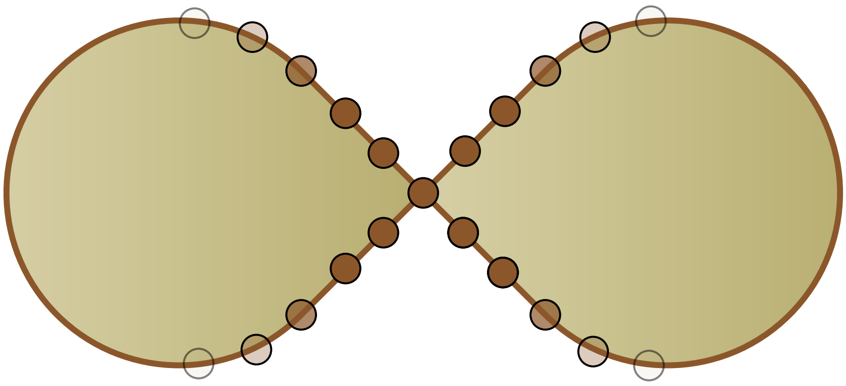

Figure 1: Lemniscate Shaped Table With Seats Marked As Brown Circles

i. Let $A = \underbrace{2, 2, \ldots, 2}_{\text{Eight 2's}}$, determine $c_{\infty}(A)$.

ii. Find a formula for $c_{\infty}(A)$ if A is exactly n copies of m.

iii. Find a formula for $c_{\infty}(A)$ with arbitrary A.

4. Let $d(n)$ denote the number of positive divisors of n.

(a) Classify all pairs of integer a and b such that $d(a) + 1 = d(ab)$

(b) Classify all pairs of integer a and b such that $d(a) + 2 = d(ab)$

(c) Classify all pairs of integer a and b such that $d(a) + d(b) = d(ab)$
(Hint: Assume the equation holds for a fixed a and b. Prove that if a is composite then something must be true of b.)

(d) Classify all pairs of integer a and b such that $d(a) + d(b) = d(ab) + 1$.

(e) Prove that there are no integer a and b such that $d(a) + d(b) \geq d(ab) + 2$.

(f) Let a and b be positive integers. Prove that $d(a^2) + d(b^2) = ab + 2$ if and only if ab is a divisor of 4.

5. For this question, I introduce the game of *Draft Nim*. The game involves two players and has two parameters, a finite sequence of natural numbers A, and a positive integer n. The game is split into two phases. First, comes the *draft phase*. The players alternate selecting entries of A until every entry has been selected, and one cannot select an entry that has already been selected. The result of this will be two sequences A_1 and A_2. Once this is done, the second phase of the game begins, the *play phase*. The play phase is a normal game of Nim on n items with the exception that each player can take a number of items only if that number is contained within the sequence that they drafted.

   For example, if $A = (1, 2, 3, 4)$ and after the draft phase $A_1 = (1, 4)$ and $A_2 = (2, 3)$ then player one can only take 1 or 4 objects while player two can only take 2 or 3 objects.

   In this version of Nim, a player loses when they can no longer make a legal move and it is their turn. This can happen when there are no items left in the pile or if the number of items left in the pile is less than every entry in their draft sequence.

   (a) Determine which player has a winning strategy for $A = (1, 2, 3, 4)$ for the following specific values of n.

      i. $n = 18$

      ii. $n = 21$

      iii. $n = 25$

   (b) If $A = (1, 2, 3, 4)$ determine which player has a winning strategy for each n.
   (Hint: Like a regular game of Nim, the solution is modular.)

   (c) Generalize your proof of the above for $A = (1, 2, \ldots, 2k)$ where k is a positive integer.

   (d) Determine which player has a winning strategy for $A = (1, 2, 3)$ and all n.

   (e) Determine which player has a winning strategy for $A = (1, 2, 3, 4, 5)$ and all n.

   (f) Determine which player has a winning strategy for $A = (1, 3, 5, 7)$ and all n.

   (g) Generalize your proof of the above for $A = (1, 3, \ldots, 4k - 1)$ where k is a positive integer.

   (h) Determine which player has a winning strategy for $A = (2, 3, 4, 5)$ and all n.

   (i) Determine which player has a winning strategy for $A = (1, 1, 2, 3)$ and all n.

   (j) Is there a sequence A of length four such that Player 2 has a winning strategy for all but a finite amount of n? If so give one such example. If not, prove that no such sequence can exist.

   (k) Determine which player has a winning strategy for $A = (1, 2, 2, 3)$ and all n.

   (l) For positive integer k, determine which player has a winning strategy for

   $$A = (1, 2, \ldots k, k, \ldots, 2k - 1),$$

   that is A includes every number between 1 and $2k - 1$ inclusive exactly once, with the exception of k which is in the sequence twice.

   (m) Choose your own sequence of four or more distinct elements. Determine which player has a winning strategy for all n

   (n) Consider an alternate version of Draft Nim for a sequence A of size three. Where player one drafts their choice of one of the three entries, and then player two gets the remaining two. Who has a winning strategy for $A = (1, 2, 3)$? What about for $A = (1, 2, 4)$?