Three Proofs For Sum Free Set Problems

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Abstract

This is a collection of similar results attained during my time conducting research during the summer of 2021. I do not feel that any of these results warranted a separate paper as they stand. However, I plan to expand upon many of these in the future. The contents of this report are by no means polished, but I do believe them to all be correct. Apart from what is in this report I also did extensive research on the standard zero-h-sum-free set problem. This will hopefully be published with a journal, but I am unsure of when I will be able to accomplish this. This report is rather bare bones, and if you have any questions regarding these results or their proofs please feel free to email me at terkja01@gettysburg.edu.

Much of the content in this report uses notation that is defined in [1], but for a quick definition one can also look **here** [3].

1 A Result Concerning Maximum Size Weak-Sum-Free sets

Here, we build on the fascinating results of Peter Francis [2]. But first, we must lay some groundwork.

Definition 1.1. *The h-fold Restricted Sum-set of a set A, written as $h\hat{}A$, is the set of all sums of h distinct elements in A.*

Definition 1.2. *A set A is weakly sum-free if $2\hat{}A \cap A = \emptyset$.*

Definition 1.3. *For some finite Abelian group G: $\mu\hat{}(G, \{2, 1\})$ will be equal to the size of the largest weakly sum-free subset of G.*

Definition 1.4. *An Abelian group's "Type" is determined as follows:*

If $|G|$ has any factor that is 2 mod 3, then G is Type I

If $|G|$ has no factors 2 mod 3, but $|G|$ is divisible by 3 then G is Type II

If $|G|$ only has factors 1 mod 3 then G is Type III

Now, we list the previous results in this topic beginning with a Theorem of Zannier

Theorem 1.5. *Theorem G.67 In [1]*

$$\mu\hat{}(\mathbb{Z}_n, \{2,1\}) = \begin{cases} \frac{n}{3}\left(\frac{1}{p}+1\right) & n \textit{ has at least 1 factor 2 mod 3, and } p \textit{ is the smallest such factor} \\ \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \textit{Otherwise} \end{cases}$$

And The Impressive Results of Peter Francis

Theorem 1.6. *Theorems 10, 11 and 14 from [2] Let G be an Abelian Group of order n with exponent κ*

- *If G is Type I*
$$\mu\hat{}(G, \{2,1\}) = \frac{n}{3}\left(\frac{1}{p}+1\right)$$
where p is the smallest factor of n congruent to 2 mod 3.
- *If G is Type II*
$$\frac{n}{3} \leq \mu\hat{}(G, \{2,1\}) \leq \frac{n}{3}+1$$
- *If G is Type III*
$$\frac{n}{\kappa} \cdot \frac{\kappa-1}{3} + 1 \leq \mu\hat{}(G, \{2,1\})$$

As well as a more specific result

Theorem 1.7. *Corollary 12 from [2] For w with no factors 2 mod 3*

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) = 3w+1$$

Another important fact that comes from Peter Francis's Paper is that he confirmed that $\mu\hat{}(\mathbb{Z}_3^3, \{2,1\}) = 9$, which will be instrumental in our proof for the following Theorem.

Theorem 1.8. *For $r \geq 3$*

$$\mu\hat{}(\mathbb{Z}_3^r, \{2,1\}) = 3^{r-1}$$

Proof. Let A be a weak sum-free set in $\mathbb{Z}_3^r$ for $r \geq 3$. Also, assume that $\mu(\mathbb{Z}_3^{r-1}, \{2,1\}) = \mu\hat{}(\mathbb{Z}_3^{r-1}, \{2,1\})$. Assume $|A| = \mu(\mathbb{Z}_3^r, \{2,1\}) + 1 = 3^{r-1}+1$. Now, we partition A into three sets A_0, A_1, and A_2 depending on the leftmost number in their ordered r-tuple representation. Note that $|A_0| \leq 3^{r-2}$ because

$\mu(\mathbb{Z}_3^{r-1}, \{2,1\}) = \mu\hat{}(\mathbb{Z}_3^{r-1}, \{2,1\})$. This means at least one of $|A_1|$ or $|A_2|$ will be greater than 3^{r-2}. Without loss of generality, we will assume that this is A_1

$$(1)\ \ |(A_0 + A_1) \cup 2\hat{}A_2| + |A_1| \leq 3^{r-1}$$
$$(2)\ \ |(A_0 + A_2) \cup 2\hat{}A_1| + |A_2| \leq 3^{r-1}$$

$$(1)\ \ |2\hat{}A_2| + |A_1| \leq 3^{r-1}$$
$$(2)\ \ |2\hat{}A_1| + |A_2| \leq 3^{r-1}$$

Utilizing Theorem D.66 in [1] we have

$$(1)\ \ 2|A_2| - 3 + |A_1| \leq 3^{r-1}$$
$$(2)\ \ 2|A_1| - 3 + |A_2| \leq 3^{r-1}$$

combining the above and simplifying to get

$$|A_1| + |A_2| - 2 \leq 2 \cdot 3^{r-2}$$

and then subtracting it from equation (2)

$$(2)\ \ |A_1| - 1 \leq 3^{r-2}$$

$$(2)\ \ |A_1| \leq 3^{r-2} + 1$$

meaning that $|A_1| = 3^{r-2} + 1$. Similarly, we arrive at $|A_2| \leq 3^{r-2} + 1$.

Overall, this gives the following bounds:

$$|A_1| = 3^{r-2} + 1$$

$$3^{r-2} - 1 \leq |A_0| \leq 3^{r-2}$$
$$3^{r-2} \leq |A_2| \leq 3^{r-2} + 1$$

Now, returning to (2) we can use Theorem D.67 in [1] and we get

$$2|A_1| - 2 + |A_2| \leq 3^{r-1}$$

If $|A_2| = 3^{r-2} + 1$, then the inequality cannot hold so we must have that $|A_2| = 3^{r-2}$, furthermore $\{2\} \times \mathbb{Z}_3^{r-1} \cap (A \cup 2\hat{}A) = \{2\} \times \mathbb{Z}_3^{r-1}$ Which gives us

$$|A_0| = 3^{r-2} \qquad |A_1| = 3^{r-2} + 1 \qquad |A_2| = 3^{r-2}$$

Now, we analyze equation (1) in a different form where we let $K = \mathrm{Stab}(A_0, A_1)$.

$$|A_0 + K| + |A_1 + K| + |A_1| - |K| \leq 3^{r-1}$$

Because $|A_1| = 3^{r-2} + 1$, and is thus aperiodic: $|A_1 + K| \geq 3^{r-2} + |K|$

$$|A_0 + K| + 2 \cdot 3^{r-2} + 1 \leq 3^{r-1}$$

$$3^{r-2} + 1 \leq 3^{r-2}$$

this is a contradiction, which gives us that

$$\mu\hat{}(\mathbb{Z}_3^r, \{2,1\}) = 3^{r-1} \implies \mu\hat{}(\mathbb{Z}_3^{r+1}, \{2,1\}) = 3^r$$

Which can be used inductively with $\mu\hat{}(\mathbb{Z}_3^3, \{2,1\}) = 9$ to prove our claim.

$\square$

Despite this section containing only one result, I am quite happy with it, and intent to investigate this topic further, and perhaps attempt to settle the Type II Case.

2 A Weighted Variation Of The Weak Sum-Free Problem

This section will first lay out many important definitions to the contents in this section as well as multiple sections in chapter 3. Then, we will present a very interesting result.

Definition 2.1. *A signed h-fold sumset of a set A, denoted as $h_{\pm}A$ is the set of all possible ways to add h not necessarily distinct elements in A. (One exception, the same element cannot be both added and subtracted in each selection of h terms)*

Definition 2.2. *A restricted signed h-fold sumset of a set A, denoted as $h\hat{_{\pm}}A$ is the set of possible ways to add and subtract h distinct elements in A.*

Definition 2.3. *A set A is signed zero-h-sum-free if $0 \notin h_{\pm}A$*

Definition 2.4. *$\tau_{\pm}(G, h)$ is equal to the maximum cardinality of a signed zero-h-sum-free subset of G.*

Definition 2.5. *A set A is sum free if $2A \cap A = \emptyset$*

Definition 2.6. *$\mu(G, \{2,1\})$ is equal to the maximum cardinality of a sum-free subset of G.*

Definition 2.7. *A set A is signed sum free if $2_{\pm}A \cap 1_{\pm}A = \emptyset$*

Definition 2.8. *$\mu_{\pm}(G, \{2,1\})$ is equal to the maximum cardinality of a signed sum-free subset of G,*

Definition 2.9. *A set A is weakly signed sum free if $2\hat{_{\pm}}A \cap 1\hat{_{\pm}}A = \emptyset$*

Definition 2.10. *$\mu\hat{_{\pm}}(G, \{2,1\})$ is equal to the maximum cardinality of a weakly signed sum-free subset of G.*

Proposition 2.11.

$$\mu_{\pm}(G, \{2,1\}) = \tau_{\pm}(G, 3) \geq v_3(\kappa, 3) \cdot \frac{n}{\kappa}$$

Proof. A is $(2,1)$-signed sum-free if and only if $0 \notin 1_{\pm}A$ and $0 \notin 3_{\pm}A$, but the latter (zero-3-signed-sum-free) requires the former, meaning a set being $(2,1)$-signed sum-free is the same as being zero-3-signed-sum-free □

Clearly it is true that $\mu\hat{}(G,\{2,1\}) \geq \mu\hat{\pm}(G,\{2,1\}) \geq \mu_{\pm}(G,\{2,1\})$ however, we can do slightly better

Lemma 2.12.

$$\mu(G,\{2,1\}) \geq \mu\hat{\pm}(G,\{2,1\}) \geq \mu_{\pm}(G,\{2,1\})$$

Proof. Let A be a weak signed sum free subset of G where $|A| > \mu(G,\{2,1\})$ With this, we would have that A cannot be sum free. This means there must exist a_0 and a_1 in A such that $a_0 = 2a_1$ (as A is certainly weakly sum free) . However, because $0 \notin A$, this would mean that $a_0 - a_1 = a_1 \in 2\hat{\pm}A$, and $a_1 \in 1\hat{\pm}A$, meaning that such a set cannot exist, proving our claim. □

The exact value of $\mu(G,\{2,1\})$ is known to be $\frac{n}{\kappa} \cdot v_1(\kappa,3)$ for all G. This is via the Results of Green and Ruzsa.

With this, we get the following result

Theorem 2.13. *For G of order n and exponent κ, we have the following*

- *For cyclic G*

$$\mu\hat{\pm}(G,\{2,1\}) = v_1(n,3)$$

- *If n is non-cyclic divisible by at least 1 number 2 mod 3*

$$\mu\hat{\pm}(G,\{2,1\}) = v_1(n,3)$$

- *If n is non-cyclic, and is divisible by 3 but no number 2 mod 3*

$$\frac{n}{3} - \frac{n}{\kappa} \leq \mu\hat{\pm}(G,\{2,1\}) \leq \frac{n}{3}$$

- *If n has only factors 1 mod 3*

$$\mu\hat{\pm}(G,\{2,1\}) = \frac{\kappa - 1}{3} \cdot \frac{n}{\kappa}$$

We are left with exact values for all G save those of Type II, however we can improve the upper bound slightly

Proposition 2.14. *For G of Type II with order n and exponent κ*

$$\mu\hat{\pm}(G,\{2,1\}) \leq \min\left\{\tau_{\pm}(G,3) + \mu\hat{\pm}(\mathbb{Z}_3^k,\{2,1\}), \frac{n}{3}\right\}$$

where $\mathbb{Z}_3^k$ is isomorphic to the largest Abelian-3 subgroup of G

Proof. Let G be a group of type II. If $\mu\hat{_\pm}(G,\{2,1\}) > \tau_\pm(G,3)$, then there must exist a weak signed $(2,1)$-sum-free $A \subset G$ such that $0 \in 3A$. However, if it is 3 distinct elements in A that add/subtract to 0 then $a+b=-c$, and if it is two then $a \pm b = \pm b$, both of which violate weak signed $(2,1)$-sum-freeness. Therefore, because $0 \notin A$, A must contain an element of order 3. Let $S = \mathrm{Ord}(G,3)$. We now let $A \cap S = A_3$.

We must have that $2\hat{_\pm}A_3 \cap A_3 = \emptyset$. Meaning, if H is the largest Abelian-3 subgroup of G, then for the canonical homomorphism $\phi(G) = H$ we have that $\phi(A_3) = B$ and B is weakly signed sum free in $H \cong \mathbb{Z}_3^k$.

Meaning that $|B| \leq \mu\hat{_\pm}(\mathbb{Z}_3^k,\{2,1\})$ giving us

$$\mu\hat{_\pm}(G,\{2,1\}) \leq \min\left\{\tau_\pm(G,3) + \mu\hat{_\pm}(\mathbb{Z}_3^k,\{2,1\}), \frac{n}{3}\right\}$$

□

We then further investigate $\mu\hat{_\pm}(\mathbb{Z}_3^r,\{2,1\})$

I have confirmed via computer program that $\mu\hat{_\pm}(\mathbb{Z}_3^3,\{2,1\}) = 4$ and $\mu\hat{_\pm}(\mathbb{Z}_3^2,\{2,1\}) = 2$

Proposition 2.15.

$$\mu\hat{_\pm}(\mathbb{Z}_3^r,\{2,1\}) = \tau\hat{_\pm}(\mathbb{Z}_3^r,[1,3])$$

Proof. Examine the conditions for a set $|A| \geq 3$ to be weakly signed sum free in $\mathbb{Z}_3^r$

We have the following for all distinct $a,b,c \in A$

- $a+b \neq c$
- $a+b+c \neq 0$ or equivalently $a+b \neq 2c$ (A is 3-AP free)
- $a \neq 2b$ or equivalently $a+b \neq 0$
- $a \neq 0$

It is seen that this is equivalent to A being weakly 3-independent, or $0 \notin [1,3]\hat{_\pm}A$.

This means

$$\mu\hat{_\pm}(\mathbb{Z}_3^r,\{2,1\}) = \tau\hat{_\pm}(\mathbb{Z}_3^r,[1,3])$$

□

With this we have the following:

Corollary 2.16.

$$\tau_\pm(G,3) \leq \mu\hat{_\pm}(G,\{2,1\}) \leq \min\left\{\tau_\pm(G,3) + \tau\hat{_\pm}(\mathbb{Z}_3^r,[1,3]), \frac{n}{3}\right\}$$

3 $\mu(G,[r,s])$

Theorem 3.1. *If G is an Abelian group with order n then*

$$\mu(G,[0,2]) \leq v_2(n,4)$$

with equality holding for cyclic G

Proof. Let A be a $[0,2]$ sum-free set, this means that $\{0\}$, A, and $2A$ are disjoint.

This means that A, $-A$ and $A-A$ are pairwise disjoint giving us

$$|G| \geq |A| + |-A| + |A-A|$$

$$|G| \geq 2|A| + 2|A+H| - |H|$$

Where $H = \mathrm{Stab}(A-A)$,

$$|G| \geq 4|A| - |H|$$

Note that for $h_1, h_2 \in H$ $(a_1+h)-(a_2+h) = a_1 - a_2 + h_3 \in A-A$ meaning that $A+H$ is $[0,2]$ sum free as well, and thus if A is maximum size, then $|A|$ is a multiple of $|H|$, thus for $d = |G|/|H|$

$$\frac{|G|}{d}\left\lfloor \frac{1}{4}(d+1) \right\rfloor \geq |A|$$

It is easily seen that

$$\max\left\{\frac{n}{d}\left\lfloor \frac{1}{4}(d+1)\right\rfloor \mid d \in D(|G|)\right\} = v_2(n,4)$$

Thus, we have proven our claim □

Corollary 3.2. *If G has order n with any factors 3 mod 4, or G has exponent κ divisible by 4*

$$\mu(G,[0,2]) = v_2(n,4)$$

From my paper with Taylor Neller we showed that

Proposition 3.3.

$$\mu(\mathbb{Z}_n,[1,s]) \geq \max\left\{\frac{n}{d}\left\lfloor \frac{d+(s-1)^2}{s^2-s+1}\right\rfloor\right\}$$

Specifically

$$\mu(\mathbb{Z}_n,[1,3]) \geq \max\left\{\frac{n}{d}\left\lfloor \frac{d+4}{7}\right\rfloor\right\}$$

Here is a table of values found for the function by Neller's computer program

Table 1: $\mu(\mathbb{Z}_n, [1, s])$

n \ s	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
7	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
8	2	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0
9	3	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
10	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0
11	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
12	4	3	2	2	1	1	1	1	1	1	0	0	0	0	0	0
13	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
14	2	2	2	2	2	1	1	1	1	1	1	1	0	0	0	0
15	5	3	3	1	1	1	1	1	1	1	1	1	1	0	0	0
16	4	4	2	2	2	2	1	1	1	1	1	1	1	1	0	0
17	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	0
18	6	3	3	3	2	2	2	1	1	1	1	1	1	1	1	1
19	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
20	5	5	4	2	2	2	2	2	1	1	1	1	1	1	1	1
21	7	3	3	3	3	1	1	1	1	1	1	1	1	1	1	1
22	4	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
23	3	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
24	8	6	4	4	3	3	2	2	2	2	1	1	1	1	1	1
25	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1
26	4	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
27	9	3	3	3	3	3	3	1	1	1	1	1	1	1	1	1
28	7	7	4	4	4	2	2	2	2	2	2	2	1	1	1	1
29	4	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
30	10	6	6	5	3	3	3	3	2	2	2	2	2	1	1	1

As you can see, equality holds with our lower bound in all known cases, but we have yet to prove this for any infinite subset of said cases with the exception of $s = 3$ shown below.

Theorem 3.4. *For an Abelian group G with order n*

$$\mu(G, [1, 3]) \leq \max\left\{ \frac{n}{d} \left\lfloor \frac{d+4}{7} \right\rfloor \mid d \in D(n) \right\}$$

With equality holding for cyclic G

Proof. For a $[1, 3]$ sum free set A we have that

$$|G| \geq |A - A| + |2A - A| + |2A|$$

Note that $\mathrm{Stab}(A - A) = \mathrm{Stab}(2A)$ as

$$h_1 + a_1 + a_2 = a_3 + a_4$$

$$h_1 + a_1 - a_4 = a_3 - a_2$$

meaning for $H_1 = \mathrm{Stab}(2A) = \mathrm{Stab}(A - A)$

$$|G| \geq 4|A| - 2|H_1| + |2A - A|$$

$$|G| \geq 4|A| - 2|H_1| + |2A| + |A| - |H_2|$$

where $H_2 = \mathrm{Stab}(2A - A)$

$$|G| \geq 7|A| - 3|H_1| - |H_2|$$

However, note that H_1 is a subgroup of H_2 which would then give us

$$|G| \geq 7|A| - 4|H_2|$$

Also, see that $H_2 = \mathrm{Stab}(3A)$ and that $A + H_2$ is also $[1, 3]$ sum free, and thus if A is maximum size then $|A|$ is a multiple of $|H_2|$ giving us

$$\mu(G, [1, 3]) \leq \max \left\{ \frac{n}{d} \left\lfloor \frac{d+4}{7} \right\rfloor \right\}$$

proving our claim. $\square$

References

[1] B. Bajnok Additive Combinatorics A Menu of Research Problems *CRC Press, Boca Raton,* 2018, p.284,

[2] P. Francis Weak (2,1) Sum Free Sets, Research Papers in Mathematics Vol. 22, Gettysburg College

[3] P. Francis
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