INEQUALITIES AND IMPLICATIONS INVOLVING THE ADJACENCY SPECTRUM OF A GRAPH

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MAY 2024

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1. Preface

This paper was written for my Math-342 class, Applied Linear Algebra. I am currently in the process of doing a more thorough literature review and turning this into an actual publication. The paper itself is an introduction to a collection of graph theoretic topics via the lens of linear algebra, and specifically: Eigenvalue analysis. We demonstrate the effectiveness of this technique by synthesizing a multitude of original results. The reader requires no graph theory knowledge but should have a good grasp of linear algebra.

2. Introduction

In this paper, we will be diving into linear algebraic graph theory. Specifically, we will be exploring how the eigenvalues of a particular matrix representation of a graph impact its graph-theoretic properties. In Section 3 we will introduce the vocabulary needed to properly understand the subsequent sections. Section 4 will provide the linear algebraic background required for the final section. In our final section, Section 5, we will discuss the impact of a graph's so-called "adjacency spectrum" on its tangible graph theoretic properties through various results provided by other authors and proofs provided by the author of this paper. Lastly, in Section 6 we discuss the so-called "smith graphs" as well as a slight generalization to the concept, for which we provide a proof for the complete classification thereof.

3. Definitions

A (simple) *graph* $\Gamma = (V, E)$ is defined by a *vertex set* V and an *edge set* E of 2-subsets of V. In this paper, we limit our view to when V is finite, and if $|V| = n$ we say that Γ has *order* n. Additionally, for any graph Γ of order n, we should note that V can be represented by any n-set of objects, and so we will follow the convention that (unless otherwise stated) $V = [n] = \{1, \ldots, n\}$. The *size* of a graph is equal to the number of edges the graph possesses, ie. the size of $\Gamma = (V, E)$ is $|E|$.

The *degree* of a vertex $v \in V$, denoted by $d(v)$, is the number of edges incident to v, that is

$$d(v) = |\{e \mid e \in E, v \in e\}|.$$

For a graph $\Gamma = (V, E)$ we define the minimal degree function, δ, to be the smallest value achieved by $d : V \longrightarrow \mathbb{Z}_{\geq 0}$ across its entire domain, ie.

$$\delta(\Gamma) = \min\{d(v) \mid v \in V\}.$$

We similarly define the maximum degree as

$$\Delta(\Gamma) = \max\{d(v) \mid v \in V\}.$$

The average degree of a graph Γ, written as $\mu(\Gamma)$, is defined as

$$\mu(\Gamma) = \frac{1}{|V|} \sum_{v \in V} d(v).$$

We say that a graph is *k-regular* if every vertex of Γ has degree k. It is seen that, for non-regular graphs,

$$\delta(\Gamma) < \mu(\Gamma) < \Delta(\Gamma).$$

The *adjacency matrix* of a graph $\Gamma = (V, E)$ with order n is the unique n by n matrix, written as A_Γ, such that

$$A_\Gamma = [a_{ij}] \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \{i, j\} \in E; \\ 0 & \{i, j\} \notin E. \end{cases}$$

One should note that A_Γ is symmetric, and contains only 0's across the main diagonal. For $i \in V$ and the corresponding standard basis vector $\mathbf{e}_i$ belonging to $\mathbb{R}^n$ we have that

$$d(i) = ||A\mathbf{e}_i||_1$$

where $|| \cdot ||_p$ is the L_p-norm.

For a matrix A the *spectrum* of A is the multiset (determined by algebraic multiplicity) of the eigenvalues of A. For example, the spectrum of

$$\begin{pmatrix} 1 & -1 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

is $\{1, 2, 2\}$. The *spectral radius* of a matrix A, sometimes written as $\rho(A)$, is equal to the magnitude of the largest eigenvalue of A. When the eigenvalues of A are real, we write the spectrum of A to be $\{\lambda_1, \ldots, \lambda_n\}$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Returning to graph theory, we say that there is a *walk* on Γ of length k from a to b (or just between a and b) for $a, b \in V$ if there exists some sequence $\{e_i\}_{1 \leq i \leq k}$ of edges and some sequence of vertices $\{v_i\}_{1 \leq i \leq k+1}$

such that $v_1 = a$, $v_{k+1} = b$, and $e_i = \{v_i, v_{i+1}\}$. It turns out that there is a walk of length k from a to b if and only if the (a, b) entry of A_Γ^k is non-zero[6]. If for all $a, b \in V$ there is a walk from a to b of length equal to some k we say that Γ is *connected*. More strictly, if there is some k for which there is a walk between all $a, b \in V$ of length k then we say that Γ is *primitive*. For clarity, let $S_\Gamma(a, b, k)$ be the statement "There exists a walk of length k from a to b." Our definitions are as follows:

$$\text{Connected: } \forall a, b \in V, \exists k \in \mathbb{N}, S_\Gamma(a, b, k),$$

and

$$\text{Primitive: } \exists k \in \mathbb{N}, \forall a, b \in V, S_\Gamma(a, b, k).$$

One can also develop linear algebraic definitions for connectivity and primitiveness. We see that a graph Γ is connected if and only if there is some $k \in \mathbb{N}$ for which $\sum_{i=1}^{k} A_\Gamma^i$ is positive[1] in which case we say that A_Γ is *irreducible*, and Γ is primitive if there is some $k \in \mathbb{N}$ such that A_Γ^k is positive, in which case we also say that A_Γ is *primitive*.

The final definitions we will are those that relate to *graph coloring*. A (proper) *coloring* of a graph $\Gamma = (V, E)$ is a partition of $V = C_1 \cup \cdots \cup C_k$ such that for any $e \in E$ with e we have that $|e \cap C_i| \neq 2$ for all i. That is, e is not a subset of any C_i. The reason this is called a coloring is that each such partition corresponds to some way in which one can color the vertices of a graph such that no vertices of the same color share an edge. Thus, each C_i is a *color* and if a graph has a proper coloring with k colors we say that the graph is k-*colorable*.

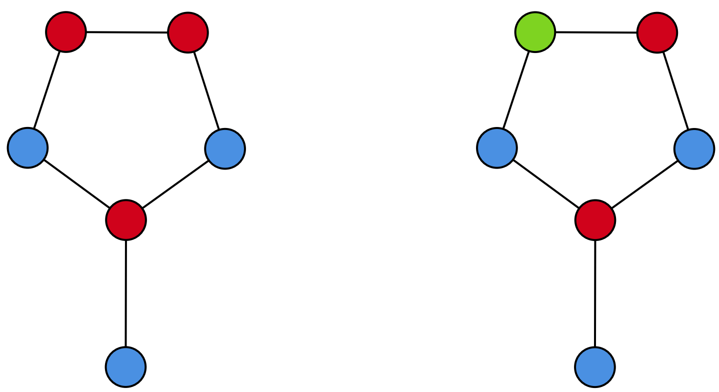

FIGURE 1. A proper (right) and an improper (left) coloring of the six-vertex graph R_5. This graph is 3-colorable but not 2-colorable.

From a linear algebraic perspective, we can similarly define colorings. For a graph Γ with order n, define some set $\mathbf{c}_1, \ldots, \mathbf{c}_k$ of binary vectors in $\mathbb{R}^n$ such that

$$\sum_{i=1}^{k} \mathbf{c}_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

It can be seen that such a set can be identified with a partition of $V = [n]$ via the rule

$$\text{the } j\text{th entry of } \mathbf{c}_i = 1 \iff j \in C_i \tag{1}$$

[1]A matrix B is said to be positive (non-negative) if every entry of B is a positive (non-negative) real number.

for $j \in [n]$. See that $(A\mathbf{c}_i)_\ell$ is non-zero if and only if the vertex ℓ is adjacent to some vertex j for which $(\mathbf{c}_i)_j$ is non-zero. For this reason, we say that a set of vectors of the aforementioned form is a (proper) coloring if for all $i \in [k]$ we have that

$$(A_\Gamma \mathbf{c}_i) \cdot \mathbf{c}_i = 0.$$

The equivalence of this definition to the graph-theoretic definition can be seen via the non-negativity of our vectors $(A_\Gamma \mathbf{c}_i) \cdot \mathbf{c}_i \neq 0$ if and only if there is some $j \in [n]$ for which the jth entries of both $A_\Gamma \mathbf{c}_i$ and $\mathbf{c}_i$ are non-zero. However, this may only occur if there is some $\ell \in [n]$ such that the ℓth entry of $\mathbf{c}_i$ is non-zero and $a_{j\ell} \neq 0$. However, since both the jth and ℓth entries of $\mathbf{c}_i$ are non-zero, this implies that $j, \ell \in C_i$, but if $a_{j\ell} \neq 0$ then we have that there is an edge between j and ℓ, and thus, $\mathbf{c}_i$ cannot be a member of a proper coloring. From this, the equivalence of our definitions via (1) follows.

It is obvious that every graph of order n is n-colorable, and no graph with any number of edges is 1-colorable. Thus, this naturally invites the question: Given a graph Γ, what is the smallest natural number k such that Γ is k-colorable? The answer to this question is given by $\chi(\Gamma)$, the chromatic number of Γ. The function $\chi(\Gamma)$ is one of the most heavily studied functions in graph theory. The famous *four color theorem* states that if Γ can be embedded in the Euclidean plane without the edges of Γ crossing each other then $\chi(\Gamma) \leq 4$, ie. you can color any two-dimensional map with four colors or less [1]. One bound on $\chi(\Gamma)$, which can be obtained far easier, is as follows.

Proposition 3.1. *For any graph Γ we have that*

$$\chi(\Gamma) \leq \Delta(\Gamma) + 1.$$

Proof. This bound is constructed as follows: Let v be a vertex with degree $\Delta(\Gamma)$ in Γ, color v and each vertex adjacent to v with $\Delta(\Gamma) + 1$ distinct colors.

Now, if a is some vertex that remains uncolored, note that it cannot have more than $\Delta(\Gamma)$ adjacent vertices, and thus it cannot be adjacent to more than $\Delta(\Gamma)$ distinct colors, and so of the existing $\Delta(\Gamma) + 1$ there exists at least one color i such that a is not adjacent to any color i vertices, and so we color a with i.

The above step can be repeated until every vertex is colored, and thus Γ is $(\Delta(\Gamma) + 1)$-colorable. ■

One can observe that this bound is sharp in the case of *complete graphs*, K_n, for all n and *cyclic graphs* (definitions below), C_n, for odd n. In fact, due to a result of Brooks [2] it is known that these two cases are the only ones for which equality in Proposition 3.1 holds.

Theorem 3.2 (Brooks' Theorem)**.** *If Γ is any graph other than a complete graph or a cyclic graph (defined below) with odd order then*

$$\chi(\Gamma) \leq \Delta(\Gamma).$$

A complete graph on n vertices, written as K_n is the unique graph where all pairs of distinct vertices are connected.

The spectrum of K_n consists of two distinct eigenvalues, $n - 1$ which has multiplicity 1 and -1 which has multiplicity $n - 1$ [3].

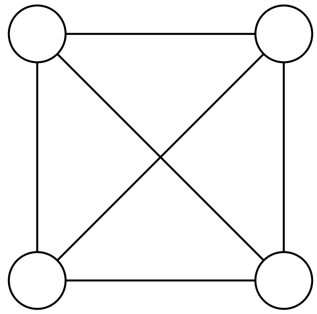

FIGURE 2. The complete graph K_4.

The cyclic graph on n vertices, denoted by C_n can be characterized by its edge set, which consists of all 2-sets of elements in $[n]$ that differ by 1 modulo n. It is seen to have the spectrum of numbers $2\cos(2\pi i/n)$ for $i \in [n]$ (see [3]).

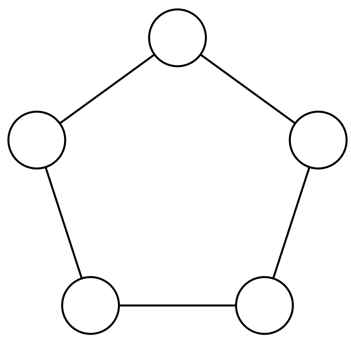

FIGURE 3. The cyclic graph C_5.

To discuss the proofs contained within this paper, we must move our discussion to the spectral properties of matrices.

4. Linear Algebraic Theorems

Recalling the linear algebraic notation from the previous section we state the following theorem.

Theorem 4.1 (Perron-Frobenius theorem). *Let A be a non-negative irreducible n by n matrix with spectrum $\lambda_1, \ldots, \lambda_n$ and spectral radius ρ. The following statements are all true.*

(1) $\rho = \lambda_1$.
(2) *There is a positive vector* $\mathbf{v}$ *such that* $\mathbf{v}$ *is an eigenvector for* λ_1.
(3) *If A is primitive then $|\lambda_i| < \lambda_1$ for all $i \neq 1$.*
(4) *If $|\lambda_i| = \lambda_1$ then λ_i is a scalar multiple of a mth root of unit for some $m \leq n$ (ie. $\lambda_1^m = \lambda_i^m$). Additionally, the spectrum of A is invariant under multiplication by $e^{2\pi i/m}$.*
(5) *If $\mathbf{u} \neq \mathbf{0}$ is a non-negative vector such that $A\mathbf{u} \leq \ell\mathbf{u}$ for some $\ell \in \mathbb{R}$, then $\mathbf{u}$ is positive and $\rho \leq \ell$.*
(6) *If $\mathbf{u}$ is a non-negative eigenvector of A then $\mathbf{u}$ is an eigenvector for λ_1.*
(7) *Any matrix B which is constructed by deleting any number rows and columns of the same index from A will have a smaller spectral radius than A.*

A proof for this version of the Perron-Frobenius Theorem may be found in [3]. The matrix B described in the above theorem is called a *principal submatrix* of A.

Because adjacency matrices are always symmetric, we also use the following result from [6].

Theorem 4.2 (Specturm of symmetric matrices). *If A is an n by n real symmetric matrix then*

(1) All the eigenvalues of A are real and
(2) $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.

Before we introduce our next theorem, we must define some vocabulary. For two decreasing sequences of real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_m$ with $m < n$ we say that b_i interlaces a_i if

$$a_i \geq b_i \geq a_{n-m+i} \quad \text{for all } i \in [m].$$

From [3] we have the following result.

Theorem 4.3 (Interlacing Eigenvalues). *If B is a principal submatrix of a symmetric matrix A then the eigenvalues of B interlace the eigenvalues of A.*

The next result we mention in this section is Parseval's Theorem. The usefulness of this result is due to the fact that our adjacency matrices are symmetric, and thus have an orthonormal basis of eigenvectors. There are many proofs for this theorem of Parseval, one of which can be found in [6]. The statement of the theorem is as follows.

Theorem 4.4 (Parseval's Identity). *If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ then for any $\mathbf{x} \in \mathbb{R}^n$ with*

$$\mathbf{x} = \sum_{i=1}^{n} s_i \mathbf{v}_i$$

we have that

$$||\mathbf{x}||_2^2 = \sum_{i=1}^{n} s_i^2.$$

Equivalently, if $P = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$ then $||\mathbf{v}||_2^2 = ||P^T\mathbf{v}||_2^2$.

We will now use this theorem to prove a fact about real symmetric matrices.

Lemma 4.5. *If A is a real symmetric n by n matrix then for any $\mathbf{v} \in \mathbb{R}^n$ we have that*

$$||A\mathbf{v}||_2 \leq ||\mathbf{v}||_2 \rho(A)$$

where $|| \cdot ||_p$ is the L_p norm.

Proof. By Theorem 4.2 we have that A is diagonalizable by a matrix P such that the columns of P form an orthonormal basis of $\mathbb{R}^n$. This gives us that

$$A = P^T D P,$$

where D is the diagonal matrix with the eigenvalues of A across said main diagonal. So, we have that

$$||A\mathbf{v}||_2 = ||P^T D P\mathbf{v}||_2.$$

Now by Theorem 4.4, the fact that P and P^T are orthogonal, and the definition of spectral radius we have that

$$\begin{aligned} ||A\mathbf{v}||_2 &= ||P^T DP\mathbf{v}||_2 \\ &= ||DP\mathbf{v}||_2 \\ &\leq \rho(A)||P\mathbf{v}||_2 \\ &= \rho(A)||\mathbf{v}||_2, \end{aligned}$$

and so our claim is proven. ■

With linear algebraic tools now at our disposal, we now move towards the focus of our paper, the intersection of these tools with graph theory.

5. Spectral Inequalities for Graphs

Just as the Perron-Frobenius theorem is one of the most important tools in spectral linear algebra, the same is true for spectral analysis in graph theory. As such, we have our own "graph theoretic" Perron-Frobenius theorem.

Theorem 5.1 (Proposition 3.1.1 in [3])**.** *For any graph Γ with adjacency matrix A having spectral radius $\rho = \lambda_1$, removing vertices or edges from Γ does not increase the spectral radius. Additionally, if is A is strongly connected we have that*

(1) Removing vertices or edges from Γ decreases ρ,
(2) λ_1 has multiplicity 1, and
(3) if A is primitive then $|\lambda_i| < \rho$ for all $i \neq 1$.

This follows from Theorem 4.1, as demonstrated in [3], and will be one of the keystones in the proofs we present. For this reason, it is stated as a theorem, as it will be more pertinent to reference this result rather than Theorem 4.1 when discussing graph-theoretic results.

We say that a subgraph $\Gamma' = (V', E')$ of $\Gamma = (V, E)$ is *induced*, if

$$E' = \{(v_1, v_2) \mid v_1, v_2 \in V, (v_1, v_2) \in E\}.$$

Equivalently, the adjacency matrix of Γ' is a principle submatrix of A_Γ, and so with this definition, we conclude the following as a result of Theorem 4.3.

Theorem 5.2. *If Γ' is an induced subgraph of Γ then the eigenvalues of Γ' interlace the eigenvalues of Γ.*

One of the simplest consequences of

Lemma 5.3 (Proposition 3.1.2 in [3])**.** *If Γ is a connected graph with spectral radius ρ. If Γ is regular then $\rho = \Delta(\Gamma)$. Otherwise*

$$\delta(\Gamma) < \mu(\Gamma) < \rho < \Delta(\Gamma).$$

If Γ is not necessarily connected, then $\delta(\Gamma) \leq \mu(\Gamma) \leq \rho \leq \Delta(\Gamma)$.

Proof. For brevity will only prove the rightmost inequality in the second assertion. A full proof can be found in [3].

Let A be the adjacency matrix of a graph Γ with order n, and let $\mathbf{1}$ be the vector in $\mathbb{R}^n$ that consists of all 1's. By Theorem 4.1 (4) we have that $A\mathbf{1} \leq \Delta(\Gamma)\mathbf{1}$ with equality if and only if $\Delta(\Gamma) = \rho$, but it is also seen that equality holds if and only if Γ is ρ-regular. ∎

Proposition 5.4 (Proposition 3.6.1 in [3]). *If Γ is a complete graph on n vertices then $\chi(\Gamma) = n$, and if Γ is an cyclic graph of odd order then $\chi(\Gamma) = 3$.*

If Γ is a graph with spectral radius ρ and Γ is not a complete graph or an odd cycle we have that

$$\chi(\Gamma) < \rho + 1.$$

Proof. Note that there must be some subgraph Γ' of Γ of minimum degree at least $\chi(\Gamma) - 1$. Via Theorem 5.2 we have that

$$\chi(\Gamma) - 1 \leq \delta(\Gamma') \leq \rho(A_{\Gamma'}) \leq \rho(A_\Gamma).$$

From this, we have that $\chi(\Gamma) \leq \rho(A_\Gamma) + 1$ with equality only if $\Gamma = \Gamma'$ via Theorem 5.1 (1). But this implies that $\delta(\Gamma) = \rho(A_\Gamma)$, which by Lemma 5.3 implies that Γ is regular, and so in this case $\rho(A_\Gamma) = \Delta(\Gamma) + 1$, but this is true only when Γ is cyclic of odd order or complete by Theorem 3.2 and so our claim is proven. ∎

A graph $\Gamma = (V, E)$ is said to be *bipartite* if there is a partition of $V = V_1 \cup V_2$ such that every edge $e \in E$ with $e = \{a, b\}$ has that (up to symmetry) $a \in V_1$ and $b \in V_2$. This partition is called a *bipartition*. There are many characterizations of bipartiteness that we characterize in Proposition 5.5. These equivalences appear throughout [3] scattered and mostly without proof. For this reason, we prove and organize these characterizations.

Proposition 5.5. *Let Γ be a (simple) graph, with spectrum $\lambda_1, \ldots, \lambda_n$. The following are equivalent*

(1) Γ is bipartite.
(2) A_Γ takes the form $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$.
(3) $-\lambda_i = \lambda_{n-i+1}$ for all $i \in [n]$.
(4) If Γ is connected then $-\lambda_1 = \lambda_n$.
(5) No cycle of any odd length is contained in Γ.
(6) If Γ is connected, then A_Γ is not primitive.
(7) $\chi(\Gamma) \leq 2$ with equality for all Γ that are not edgeless.

Proof. Let Γ be bipartite of order n with bipartition $V = V_1 \cup V_2$ such that $V = [m]$ with $m \leq n$. Since Γ is bipartite if we let $A_\Gamma = (a_{ij})$ it is seen that if i and j are both less than or equal to m then $a_{ij} = 0$. Similarly, if both i and j are greater than m then we also have that $a_{ij} = 0$. Thus, we have that $A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. However, because Γ is not a directed graph, we have that $A^T = A$, and so $C = B^T$. Thus, we have that (1) implies (2).

Similarly, if we have $A_\Gamma = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ such that B has m columns then it is seen that if $e \in E$ with $e = \{a, b\}$ then we must have either that $a \leq m$ and $b > m$ or $a > m$ and $b \geq m$. Thus, the partition $V = V_1 \cup V_2$ with $V_1 = [m]$ and $V_2 = [m + 1, n]$ is a bipartition of Γ, and so Γ is bipartite. Thus, it is seen that (2) implies (1), and so they are equivalent.

Now, see that if Γ is bipartite with adjacency matrix $A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ with eigenvector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and associated eigenvalue λ it is seen that $\mathbf{v} = \begin{pmatrix} a \\ -b \end{pmatrix}$ is also an eigenvector with respect to A_Γ but with associated eigenvalue $-\lambda$, and this generalizes to the entire spectrum of A_Γ, and so we have that (1) implies (3) as well as (4).

Note that if we have $\lambda_1 = -\lambda_n$, then by Theorem 4.1, Γ cannot be primitive, and (4) so implies (6). Additionally since (1) implies (4) transitively, it is seen that (1) implies (6).

Let us say that a connected graph $\Gamma = (V, E)$ is k-primitive if A_Γ^k is positive. By definition, if Γ is primitive, then it must be k-primitive for some $k \in \mathbb{N}$. Since A_Γ is connected, it has no zero columns, and so A_Γ^{k+1} is positive. So, we have that if Γ is primitive then there exists some odd natural number k_0 such that Γ is k_0-primitive. Thus, for any vertex v^* there is a walk of on length k_0 from v^* to itself. Thus, there exists some walk of minimum odd length $k_{\min}$ from v^* to v^*. That is, if $\ell \leq k_{\min}$ and there exists a walk W of odd length from v^* to v^* then $\ell = k_{\min}$. Since W is minimal in length with respect to odd walks, it cannot contain a cycle of even length, as such a cycle can be contracted down to a single point while preserving the parity of the walk's length. Thus, one of two things must be true either (i) every vertex in W except the start/end is distinct, and thus W itself is a cycle or W contains a cycle, which by the above must be of odd length. Either way, the implication that Γ contains a cycle of odd length arises and so we have that (5) implies (6).

Now, observe that if $\Gamma = (V, E)$ is connected, then for all sufficiently large $N \in \mathbb{N}$, there exists a path of length N between any pair of vertices in V as long as N is divisible by the greatest common divisor of all cycle lengths found in Γ. Additionally, since Γ is connected not directed, as long as Γ contains more than one vertex, a 2-cycle always exists by going from a starting vertex to a different vertex and back to the start again. Thus, if Γ contains an odd cycle, then it is primitive. From this, we have that (6) implies (5).

Now, we prove that if no odd cycle lies in Γ, then it is bipartite. I claim that the parity of the length of a walk between any pair of connected vertices v_1 and v_2 is consistent regardless of the walk itself. Indeed, there are two walks of length ℓ_1 and ℓ_2 respectively each with different parity, both between v_1 and v_2 then there exists a length $\ell_1 + \ell_2$ path from v_1 to itself. Since ℓ_1 and ℓ_2 have different parities, there exists an odd length walk from v_1 to itself, and by the same argument as (5) $\implies$ (6), Γ must contain an odd cycle, which is a contradiction to the original assumption. Thus, all paths between a fixed connected vertex pair have the same parity. Now for any maximal connected subgraph $\Gamma' = (V', E')$ of Γ we do as follows. First, select an arbitrary vertex v of Γ'. By the above, the set of vertices V_1' which possess a path of odd length from v is disjoint from the set of vertices V_2' which possess a path of even length from v, and because Γ' is connected it follows that $V_1' \cup V_2'$ is a partition of Γ'. This partition is also seen to be a valid bipartition as no two vertices in V_i may be adjacent as if they were then that would imply that there is some x such that x and $x + 1$ have the same parity, which is impossible. Thus, we have that (5) implies (1).

We now show that (3) implies (1). Let Γ be a graph with a spectrum $\lambda_1 \geq \cdots \geq \lambda_n$ satisfying the condition in (3). Note that, for odd k, the spectrum of A_Γ^k is $\lambda_1^k \geq \cdots \geq \lambda_n^k$. However, because k is odd, we have that $(-\lambda_i)^k = -\lambda_i^k$, and so we have that

$$\operatorname{tr}(A_\Gamma^k) = \lambda_1^k + \cdots + \lambda_n^k = 0.$$

From this, and the fact that A_Γ^k is non-negative we have that the diagonal of A_Γ^k must be all zeroes, and thus Γ contains no k-cycles for any odd k. Thus, (3) implies (5), and because (5) implies (1), we have that (1) and (3) are equivalent.

Now, if we let Γ be connected with $\lambda_1 + \lambda_n = 0$, then by Theorem 4.1 it follows that the entire spectrum of Γ is invariant under multiplication by −1, and so (3) is seen to be equivalent to (4).

Finally, see that $\chi(\Gamma) = 1$ if and only if Γ is edgeless, in which case Γ is trivially bipartite. In general, we observe that, via the definitions, bipartite is equivalent to 2-colorable, as the bipartition is itself the coloring. This gives us the final equivalence of (1) and (7).

■

The *complete bipartite* graph on m and n vertices , written as $K_{m,n}$ is the unique bipartite graph on $m+n$ vertices such that the bipartite partition $V = V_1 \cup V_2$ has $|V_1| = m$ and $|V_2| = n$ with

$$E = \{\{a, b\} \mid a \in V_1, b \in V_2\}.$$

One can observe that the spectrum of a complete bipartite graph consists of the eigenvalues $\pm\sqrt{mn}$, each with multiplicity 1 as well as 0 with multiplicity $m + n - 2$[3].

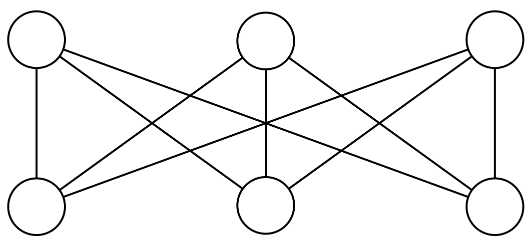

FIGURE 4. The complete bipartite graph $K_{3,3}$

It turns out, that the property $\text{rank}(A_\Gamma) = 2$ is a property that no other graphs possess. We prove this in Lemma 5.6, below.

Lemma 5.6. *If* Γ *is a connected graph we have that* $\text{rank}(A_\Gamma) = 2$ *if and only if* Γ *is complete bipartite.*

Proof. The if direction swiftly follows from the fact that the adjacency matrix of $K_{m,n}$ is

$$A_{K_{m,n}} = \begin{pmatrix} 0 & 1_{m\times n} \\ 1_{n\times m} & 0 \end{pmatrix}$$

which has exactly two distinct columns, both of which are non-zero.

Note that, if Γ′ is an induced subgraph of Γ then

$$\text{rank}(A_\Gamma) \geq \text{rank}(A_{\Gamma'}).$$

By Proposition 5.5, if Γ is not bipartite, it must contain an odd cycle. In any graph Γ with an odd cycle subgraph, it is true that any minimum length odd cycle is an induced subgraph of Γ. This can be seen as for any given minimum length odd cycle subgraph, it cannot contain a chord connecting any pair of its vertices, as that chord would yield a smaller odd cycle. Indeed, it can be seen that for odd n, $\text{rank}(A_{C_n}) = n$. Thus, Γ is bipartite. Now, note that if Γ is bipartite with more than one edge (If Γ has only one edge but is connected then it is obvious that $\Gamma = K_{1,1}$, and thus is complete bipartite.) but is not complete bipartite, then there exists some pair of edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ such that v_1 is not adjacent to either of v_3 or v_4. This implies that Γ has an induced subgraph Γ' which has adjacency matrix equal to either

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In either case, $\operatorname{rank}(A_{\Gamma'}) = 4$, and so Γ must be complete bipartite. ∎

If a graph Γ is of the form $K_{1,d}$ then we say that Γ is a *star graph*.

As for additional bounds on $\rho(\Gamma)$, we have the following known bound (see [5]) for which we characterize equality.

Proposition 5.7. *For a connected graph Γ with maximum degree Δ and adjacency matrix A that has spectral radius ρ we have that*

$$\sqrt{\Delta} \leq \rho$$

with equality if and only if Γ is a star graph.

Proof. Let $\mathbf{v}$ be the vector with 1 at the index of some vertex with maximum degree. Via Lemma 4.5 we have that

$$\sqrt{\Delta} = ||A\mathbf{v}||_2 \leq \rho||\mathbf{v}||_2 = \rho(A).$$

Now, we demonstrate the conditions for equality holding in our claim. Assume that $\sqrt{\Delta} = \rho$.

Since A is symmetric, there exists an orthonormal basis of $\mathbb{R}^n$, $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ consisting only of eigenvectors of A. Let $\mathbf{v} = s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n$. Since $\mathbf{u}_i$ are all eigenvectors of A we have that

$$A\mathbf{v} = \sum_{i=1}^{n} s_i \lambda_i \mathbf{u}_i.$$

Via Theorem 4.4 we must have that

$$||\mathbf{v}|| = 1 = \sum_{i=1}^{n} s_i^2$$

and

$$||A\mathbf{v}||^2 = \rho^2 = \sum_{i=1}^{n} (\lambda_i s_i)^2 \leq \rho^2 \sum_{i=1}^{n} s_i^2.$$

See that equality holds in the above if and only if $|\lambda_i| = \rho$ for all $s_i \neq 0$, and since $1 = \sum_{i=1}^{n} s_i^2$ equality clearly does hold and so s_i is zero for all i except for those with $\lambda_i = \pm\rho$. Now, by Proposition 5.5 we have that either Γ is bipartite, in which case the spectrum of Γ is $\{\rho, 0, \ldots, 0, -\rho\}$ or that Γ is primitive.

In the latter case, we would have that $\mathbf{v} = \mathbf{u}_1$, but this cannot be as $\mathbf{u}_1$ is strictly positive via Theorem 4.1 and $\mathbf{v}_1$ is not strictly positive.

Thus, Γ is bipartite and the spectrum of Γ must be exactly $\{\rho, 0, \ldots, 0, -\rho\}$. Since 0 has multiplicity $n-2$ and Γ is connected it is seen that Γ is indeed complete bipartite via Lemma 5.6, and so we have that $\rho = \sqrt{\Delta(n-\Delta)}$, but we have that $\rho = \Delta$, and so $\Delta = n-1$, implying that Γ is a star graph. If Γ is in fact a star graph it is easy to see that it satisfies $\Delta = \rho^2$, and so our claim is proven. ∎

This result can reformed into the following bound.

Corollary 5.8. *If Γ is a connected graph with order n, maximum degree Δ, and spectral radius ρ we have that*

$$\frac{\Delta}{\sqrt{n-1}} \leq \rho$$

with equality if and only if Γ is a star graph.

Proof. Note that

$$\Delta \leq n-1,$$

and from this we get that

$$\Delta^2 \leq (n-1)\Delta.$$

Via Proposition 5.7 we now have that

$$\Delta \leq \sqrt{n-1}\sqrt{\Delta} \leq \rho\sqrt{n-1},$$

and so

$$\Delta/\sqrt{n-1} \leq \rho.$$

■

While the result in Proposition 5.7 is as good or better than the bound from Corollary 5.8, I find that both results are enlightening, at least slightly as they express how different tangible quantities may affect the spectral radius (a relatively intangible quantity). In general, bounds on eigenvalues relative to the vertex degrees seem rather interesting, and of specific interest to myself is the following.

Question 5.9. Is there a nice characterization for graphs with maximum degree Δ with spectral radius ρ satisfying

$$c\sqrt{\Delta} \geq \rho \geq \sqrt{\Delta}$$

for some fixed $c \in \mathbb{R}$?

More specifically, one may want to tackle this question:

Question 5.10. Are there finitely many connected graphs with maximum degree Δ and spectral radius ρ satisfying

$$\sqrt{\Delta+1} \geq \rho \geq \sqrt{\Delta},$$

and if so, what are they?

In regards to Question 5.10, the author believes that there are finitely many such graph, that they are all bipartite. For the case of $\Delta = 3$ one could utilize Theorem 6.2 in order to obtain such a classification, notably, though, the answer given by this result is that there are infinitely many such graphs. Any proof for $\Delta \geq 4$ remains elusive.

Proposition 5.11. *If Γ is a connected graph with smallest eigenvalue λ_n we have that*

$$\lambda_n \leq -\sqrt{2}$$

as long Γ is not complete, if Γ is complete then $\lambda_n = -1$.

Proof. For the if direction it is seen that equality does hold if Γ is complete. Consider some graph of order n, Γ. It suffices to assume Γ is not complete. By the connectivity of Γ we have that P_3 is an induced subgraph of Γ. Thus, by Theorem 5.2, we have that $-\sqrt{2} \geq \lambda_n$ as $-\sqrt{2}$ is the smallest eigenvalue of P_3[3], and so our claim follows.

■

While this result is certainly interesting, it is not quite substantial. However, this next result is much more enticing to discuss.

6. A Smith-like Spectral Characterization for Graphs

Before moving into the promised characterizations, we will define a couple classes of graphs. The *path graph* $P_n = (V, E)$ on n vertices is the unique graph with

$$E = \{\{i, i+1\} \mid i \in [n-1]\}.$$

Its spectrum consists of the numbers of the form $2\cos\left(\frac{\pi i}{n+1}\right)$ for $i \in [n]$ [3].

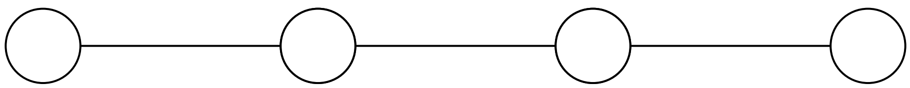

Figure 5. The path graph P_4.

The *racquet graph*, R_n is a $n+1$ vertex graph equal to a cycle graph with a single additional vertex attached to the cycle at one vertex. It has the adjacency matrix

$$A = (a_{ij}) \quad \text{where} \quad a_{ij} = \begin{cases} 1 & i, j \leq n \text{ and } |i-j| \text{ is } 1 \bmod n; \\ 1 & \{i, j\} = \{n, n+1\}; \\ 0 & \text{otherwise.} \end{cases}$$

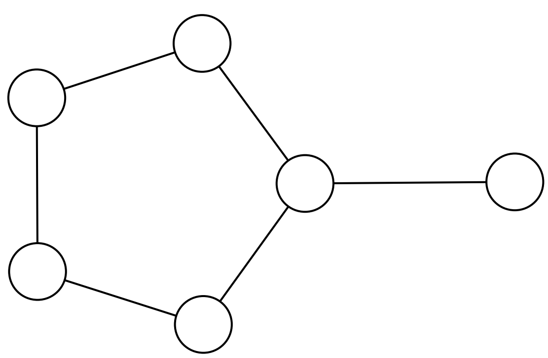

Figure 6. The Racquet Graph R_5.

The spectral properties of R_n are not well known, but we will prove the following lemma which will be of great importance in the main result of this section.

Lemma 6.1. *When $\lambda_1 - \lambda_{n+1} \leq 4$ for eigenvalues of R_{n+1}, we have that $n = 3$ or $n = 5$.*

Proof. Let $\mathbf{v}$ be the all-ones vector in $\mathbb{R}^{n+1}$. By Lemma 4.5

$$||A\mathbf{v}|| \leq \lambda_1 ||\mathbf{v}||$$

but since $||\mathbf{v}|| = \sqrt{n+1}$ and $||A\mathbf{v}|| = \left(1^2 + 3^2 + (n-1)2^2\right)^{1/2} = \sqrt{4n+6}$, and so

$$\sqrt{\frac{4n+6}{n+1}} \leq \lambda_1.$$

So, if $\lambda_1 - \lambda_{n+1} \leq 4$ then we have that

$$\sqrt{\frac{4n+6}{n+1}} - \lambda_{n+1} \leq 4,$$

but by Theorem 5.2 and the fact that P_n is an induced subgraph of R_n, this gives us that

$$\sqrt{\frac{4n+6}{n+1}} + 2\cos(\pi/(n+1)) \leq 4.$$

It is well known that $\cos(\theta) \geq 1 - \frac{\theta^2}{2}$, and so we have that

$$\sqrt{\frac{4n+6}{n+1}} \leq 2 + \frac{\pi^2}{(n+1^2)}.$$

We may square both sides to get

$$4 + \frac{2}{n+1} \leq 4 + \frac{4\pi^2}{(n+1)^2} + \frac{\pi^4}{(n+1)^4}$$

Letting $X = n+1$, this is equivalent to

$$2X^3 - 4\pi^2 X^2 - \pi^4 \leq 0.$$

Via analyzing the roots of this polynomial, it is seen that this is true when $X \leq 20$.

Thus, we must have that $n \leq 21$ if $\lambda_1 - \lambda_n \leq 4$. Computational analysis on the adjacency matrix of R_n for $n \leq 21$ reveals that the only values of n that this inequality holds for are $n = 3$ and $n = 5$. ∎

In 1970, Smith [7] determined all graphs that have a spectral radius less than or equal to 2. For this reason, we call graphs with this property *Smith Graphs*.

Theorem 6.2 (Characterization of Smith Graphs). *If $\rho(\Gamma) \leq 2$ then Γ must be a subgraph of either a cyclic graph, D_n for $n \geq 4$, E_6, E_7, or E_8 (see Appendix A for definitions of these graphs).*

These graphs are not only interesting from a spectral standpoint but within Lie Groups as they can all be identified with Dynkin Diagrams of finite Coxeter groups, and are of great importance to the classification of finite simple groups [3].

It should be noted that Theorem 4.1, if the largest eigenvalue of Γ is less than or equal to 2 then we have that $\lambda_1 - \lambda_n \leq 4$. However, the converse of this statement is seen to not be true via the witness K_4.

Theorem 6.3. *If Γ is a connected graph with spectrum $\lambda_1, \ldots, \lambda_n$ and $\lambda_1 - \lambda_n \leq 4$ then either Γ is a connected subgraph of a Smith graph or it is one of K_4, R_3, R_5, $\mathcal{A}$, or $\mathcal{J}$ (see Figure 9 for the latter two).*

Proof. $\lambda_1 - \lambda_n \leq 4$ implies either that $\lambda_1 \leq 2$ or that $\lambda_1 > 2$ and $\lambda_1 > -2$. If $\lambda_1 \leq 2$, by Theorem 6.2 it follows that Γ is a connected subgraph of a Smith graph. So, it suffices to consider when $\lambda_1 > 2$. In this case, since $\lambda_1 > 2$ we have that Γ cannot be bipartite by Proposition 5.5 (4).

Since Γ is not bipartite it contains a minimal odd cycle of length t, which via minimality implies that C_t is an induced subgraph of Γ. As long as Γ itself is not cyclic then there must be some other vertex v of Γ which is itself adjacent to some vertex in the minimal cyclic subgraph, but itself does not lie in the cycle. Additionally, we should note that if v is adjacent to more than a single vertex in C_t then there exists a cycle in Γ if length at most $(t+3)/2$, and by the minimality of t we must have that $(t+3)/2 \leq t$, and so we must have that at least one of the following holds:

- R_t is an induced subgraph of Γ,
- K_4 is an induced subgraph of Γ,
- or $\mathcal{D}$ (See Figure 7) is an induced subgraph of Γ.

Via Theorem 5.1 and Theorem 5.2, if Γ' is one of the induced subgraphs described above having largest and smallest eigenvalues μ_1 and μ_n respectively then we must have that $\mu_1 - \mu_n \leq 4$ with equality if and only if $\Gamma = \Gamma'$. With this, we see that if $\Gamma' = \mathcal{D}$, then $\mu_1 - \mu_n > 4$, and so this is not a possibility. If $\Gamma' = K_4$ then $\mu_1 - \mu_n = 4$ and we must have that $\Gamma' = \Gamma = K_4$.

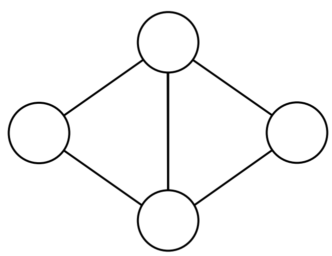

Figure 7. The diamond graph $\mathcal{D}$.

So, if $\Gamma \neq K_4$ then we must have that R_t is an induced subgraph of Γ. Thus, by Theorem 5.2 and Lemma 6.1, we have that Γ has either a triangle or a 5-cycle.

Assume Γ has a 5-cycle, but $\Gamma \neq R_5$. Thus, there exists some vertex v^* in Γ that is adjacent to the induced R_5 subgraph, but itself is not a member of it. Thus, we see that Γ must have one of the three subgraphs in Figure 8. Letting μ_1 be the largest eigenvalue of the seven vertex graph in question, and let ν_6 be the smallest eigenvalue of R_5. Via Theorem 4.1 and Theorem 5.2 we have that

$$4 - \mu_1 \leq \lambda_n \leq \nu_6.$$

Checking all each value of μ_1, we see that this is never true, and so if Γ has a 5-cycle then $\Gamma = R_5$.

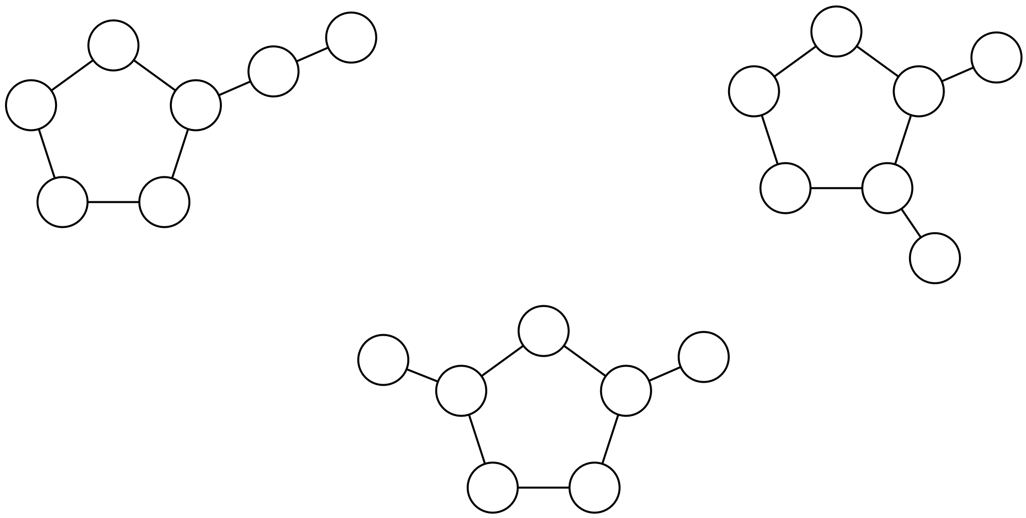

FIGURE 8. Potential 5-cycle subgraphs of Γ

A similar procedure is performed on Γ with a three-cycle, and it is seen that Γ must have an induced subgraph that is one of the two graphs in Figure 9. Upon repeating this process on each of $\mathcal{A}$ and $\mathcal{J}$, we get that no vertices can be added without violating $\lambda_1 - \lambda_n \leq 4$.

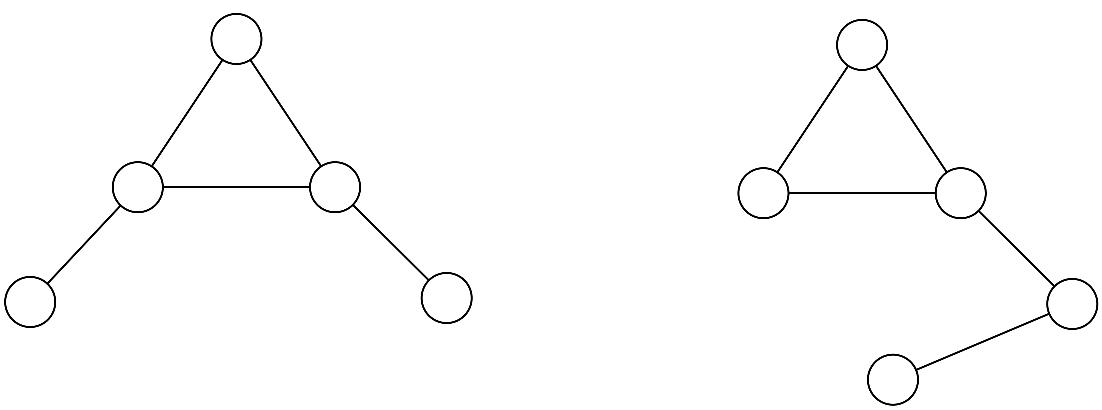

FIGURE 9. Potential 3-cycle induced subgraphs. The graph $\mathcal{A}$ (left) and the graph $\mathcal{J}$ (right).

■

The actual largest and smallest eigenvalues of the five non-smith graphs satisfying $\lambda_1 - \lambda_n \leq 4$ are cataloged in the table below.

Γ	λ_1	λ_n	$\lambda_1 - \lambda_n$
K_4	3	-1	4
R_3	≈ 2.17	≈ -1.48	≈ 3.65
$\mathcal{A}$	$(1+\sqrt{13})/2$	$-(1+\sqrt{5})/2$	$(2+\sqrt{13}+\sqrt{5})/2 \approx 3.92$
$\mathcal{J}$	≈ 2.21	≈ -1.68	≈ 3.71
R_5	≈ 2.11	≈ -1.86	≈ 3.97

Table 1. The actual value of $\lambda_1 - \lambda_n$ for the five graphs classified by Theorem 6.3.

Conjecture 6.4. *There are a finite number of connected non-smith graphs that satisfy* $\lambda_1 - \lambda_n \leq 3\sqrt{2}$.

Using the search feature on houseofgraphs.org, (see [4]) we find that at least 41 graphs in addition to the five from Theorem 6.3 satisfy this criterion. To prove this, one could likely follow a similar process to the proof of Theorem 6.3, but two major hurdles would need to be overcome. The first of these obstacles is the fact that there are non-smith bipartite graphs that meet this criterion, this adds a whole other dimension to this theoretic proof. The second obstacle one would need to overcome is to determine a much better bound on this "spectral length" for Racquet graphs, as the one from Lemma 6.1 is not very good. Or alternatively, an entirely different proof route may be optimal. Another conjecture that I have that seems to be true is tangentially related to Theorem 6.2, but concerns a different classification entirely.

Conjecture 6.5. *If* Γ *is regular,* $\lambda_n > -2$ *and* $\lambda_1 > 2$ *then* Γ *is complete.*

Appendix A. Smith Graphs

This appendix serves to illustratively define the graphs mentioned in Theorem 6.2.

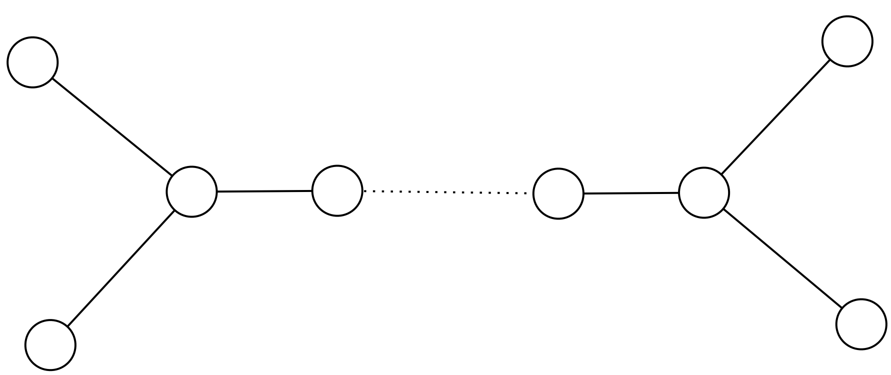

Figure 10. The graph D_n has $n+1$ vertices and is defined for $n \geq 4$.

Apart from the family D_n, we provide visual definitions 3 more individual graphs.

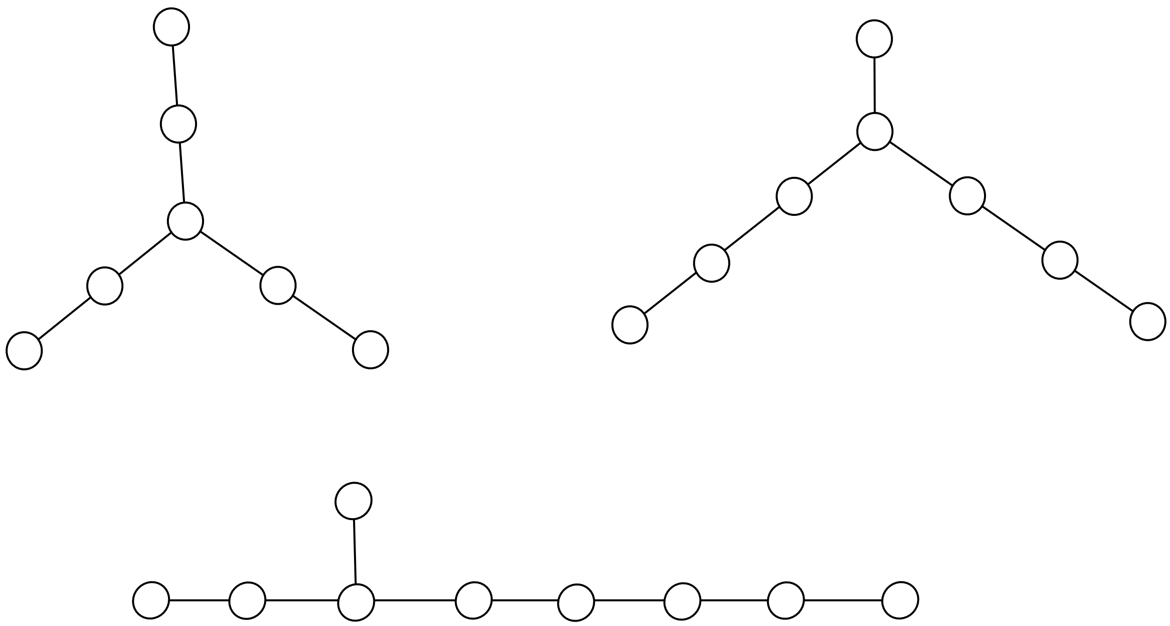

FIGURE 11. The graphs E_6 (left), E_7 (right), and E_8 (bottom).

The spectral radius of all of D_n, E_6, E_7, and E_8 have spectral radius of exactly 2, and along with C_n (the cyclic graph on n vertices) form the complete set of graphs with spectral radius 2 [7].

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