Challenge Problems 3

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Problem Index

Problem	1	2	3	4	5	6
Difficulty/10	7.5±1	7	6	2	2.5	1.5
Category	NT	CO	NT	GM	PB	NT

Key:

- NT: Number Theory
- CO: Combinatorics
- PB: Probability
- GM: Geometry

1. Let $\mathbb{P}$ be the set of positive integer primes. The set of positive integers k that satisfy $|\mathbf{P}_k| = 1$ where
$$\mathbf{P}_k = \{a \mid a \in \mathbb{Z}_k, a \equiv p^2\ (k) \text{ for infinitely many } p \in \mathbb{P}\}$$
is equivalent to the set of divisors of positive divisors of 24.[1]
   (a) Classify when $|\mathbf{P}_k| \leq 3$.
   (b) Now let
   $$\mathbf{P}'_k = \{a \mid a \in \mathbb{Z}_k, a \equiv p^3\ (k) \text{ for infinitely many } p \in \mathbb{P}\}.$$
   Show that if $k \geq 3$ then $|\mathbf{P}'_k| > 1$.
   (c) Find when $|\mathbf{P}'_k| \leq 4$

2. How many possible ways are there to move from the origin, $(0, 0)$, to $(5, 5)$ with the movement options $\{(2, 0), (1, 0), (0, 1), (0, 2)\}$?

3. Let k be an integer greater than 2. Find a set of necessary and sufficient conditions on k such that there are exactly three distinct integer values of n between 1 and k exclusive such that there is an integer m where
$$n^2 - 1 = mk.$$

4. Let a and b be known lengths of a triangle's edges, and let A be the known angle opposite of the side a. Find a set of conditions on A, a, and b such that all triangles that meet these conditions with the parameters described above are congruent.

5. Suppose a \$1 slot machine has a $\frac{1}{10^n}$% chance of resulting in a jackpot for some integer $n \geq 1$. At this particular slot machine every 10 spins you are awarded a free spin. Let $P(n)$ be the probability that you win at least one jackpot after spending 10^n dollars. Find $\lim_{n\to\infty} P(n)$.

6. Prove that every prime number other than 2 and 5 divides an infinite amount of numbers of the form $11....11$ (11, 111, 1111, etc.).

[1] My inspiration for this problem.