Challenge Problems

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Key:

- NT: Number Theory
- AL: Algebra
- AN: Analysis
- CO: Combinatorics
- GM: Geometry

1. Prove that the equation
$$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + 1 = 2^n$$
only has finitely many solutions for positive integers m and n.

2. Given some Gaussian integer $a \in \mathbb{Z}[i]$ show that the group of Gaussian integers mod a is cyclic under addition if a is a Gaussian prime with both real and imaginary components non-zero.

3. Show that the quantity
$$a = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{\cdots}}}}}$$
exists, ie. the above nested radical converges onto a real number.

4. Prove that in any set A of 7 numbers with different remainders when divided by 15, there exists $a, b, c \in A$ such that $a + b + c$ is divisible by 15. (a, b, c need not be distinct).

5. Find all positive integer solutions to
$$a^2 - b^2 = 2^k$$
where both a and b are odd.

6. Prove that the square is the only regular polygon with side length 1 that has integer area.

7. Prove that
$$1^3 + 2^3 + \cdots + n^3$$
is never prime.

8. Determine when the polynomial $f(x) = x^3 + 2abx^2 + ax$ has only real rational roots for $a, b \in \mathbb{Z}$.

9. Equations of the form
$$x^3 + y^3 + ax^2 + by^2 + cxy = 0$$
sometimes have two self intersections, find the conditions for this occurring (in terms of an equation or inequality containing only a, b and c as variables) and the points of self intersection in terms of a, b and c).

10. Let p be some prime not equal to 2 or 5. Prove that $\frac{1}{p}$ cannot be represented as a terminating decimal in base 10, and prove that ℓ_{10}, the period of the decimal's repeating cycle, divides $p - 1$.